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W. J. GREENSTREET, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
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The Mathematical Association.

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RECENT DEVELOPMENTS IN INVARIANT THEORY.

BY PROF. H. W. TURNBULL, M.A.

I HAVE been asked to write a short account of the present-day work in the algebra of invariants for the benefit of those who have not made a special study of the subject. A good summary account can be found in the various continental works of reference, and particularly in the *Encyklopädie der Mathematischen Wissenschaften*, Band III. Teil iv. 7, written by Dr. R. Weitzenböck, and brought up to about the date 1920 with tolerable completeness, both in subject-matter and in references. Nothing has, however, been published in the last thirty years quite so good and readable as Meyer's *Bericht*,* which reviewed the first fifty years of progress from the time when Boole inaugurated the theory in the *Cambridge Mathematical Journal*, iii. pp. 1-20, of the year 1841. As things have turned out, and especially because of the impossibility of buying separate parts of the *Encyklopädie* without embarking on the whole section (Geometry in this particular case), it seems a pity that we have no machinery in England for the systematic issue of corresponding reviews, perhaps in the form of this *Bericht*, to cover by degrees the field of present-day mathematics.

At the risk of saying what is perfectly well known, I shall first explain what is meant by an invariant. It is something which persists and can be recognized amid the change and flux which goes on all around. To attach a significance to certain functions, called invariants in algebra, is to do what we are all doing consciously or unconsciously in everyday life. It is to sort out and tidy up what is at first sight a medley of confusion and change. It is an effort to recognize what, because of its form or colour or meaning or otherwise, is important and significant in what is only trivial or ephemeral. A simple instance of failing in this is provided by the poll-man at Cambridge, who learnt perfectly how to factorize a^2-b^2 but was floored because the examiner unkindly asked for the factors of p^2-q^2. Here we have all the ingredients of an invariant theory. Certain numbers called a and b exist and a certain property of them. A change is made—in this case the simplest possible binary linear transformation ; for a we write p and for b we write q—

* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, i. 1892. (G. Reimer, Berlin.) Issued separately. The Report can also be had, slightly abridged, in a French translation.

and in spite of this the property emerges unchanged. More generally let us suppose that x is a function of y, $x=f(y)$, and that a function ϕ exists such that

$$\phi(x)=\phi(y)$$

identically. Then ϕ is said to be an invariant of the transformation indicated by $x=f(y)$.

Still more generally, suppose that there is a set (x) of n numbers x_1, x_2, x_3, ... x_n, each of which is a function of another set $y_1, y_2, y_3, \dots y_n$. Let this be indicated by the statement

$$\left.\begin{array}{l} x_i=f_i(y_1, y_2, \dots y_n), \\ i=1, 2, 3, \dots n. \end{array}\right\} \qquad \text{.............................(1)}$$

Then if a function ϕ exists such that

$$\phi(x_1, \dots x_n)=\phi(y_1, \dots y_n) \qquad \text{.............................(2)}$$

identically, ϕ is said to be an invariant of the transformation (1).

Put in this form it will be seen that the theory should have considerable interest for physicists as well as for algebraists. For we may reverse the usual order wherein the functions f are given and ϕ has to be found, by starting with ϕ and trying to find the functions f. If, for instance, we assume $n=4$ and

$$\phi(x_1, x_2, x_3, x_4)=x_1^2+x_2^2+x_3^2-c^2x_4^2,$$

and then proceed to ask what is the possible set of four functions f, and, in particular, *linear* functions f which allow conditions (1) and (2) to hold, we immediately touch upon the theory of restricted relativity. For the answer to this is precisely the algebraic expression for the possible linear transformations of this theory.*

A little thought on the possibilities of equations (1) and (2) for small values of n, $=1, 2, 3$ say, will reveal the fundamental importance of the invariant idea in all such branches of mathematics as analytical geometry, the method of vectors, statics, dynamics, and differential geometry.

Next suppose that we consider the equations (1) as a transformation process T by which we substitute the set (y) for the set (x), then it is convenient to have a symbol for this. We simply write

$$T: \quad x \to y,$$

which in turn suggests a reverse process

$$T^{-1}: \quad y \to x.$$

It may be said at once that the existence of this reversibility is a *sine quâ non* of the theory. Further, the success of the theory requires the study of a *group* of such transformations. Thus if S is a new set of equations giving $y_1, y_2, \dots y_n$ in terms of $z_1, z_2, \dots z_n$, and written

$$S: \quad y \to z,$$

then the direct step from x to z, which is written

$$TS: \quad x \to z,$$

must be reckoned as a definite single transformation, T' say, in the same category as T or S. If we restrict the functions f in (1) all to be linear functions, and write

$$\left.\begin{array}{l} x_i=c_{i1}y_1+c_{i2}y_2+\dots+c_{in}y_n, \\ i=1, 2, 3, \dots n, \end{array}\right\} \qquad \text{.............................(3)}$$

we have the general homogeneous linear transformation of n variables x. Clearly a linear transformation $x \to y$, followed by another linear one $y \to z$,

* A good elementary account of this is in the recent book by E. Study, *Theorie der Invarianten auf Grund der Vectorenrechnung*. (Braunschweig, 1923.)

is equivalent to a linear transformation from $x \rightarrow z$. So here is an example of a *group*. Each such group of transformations, whether linear or more elaborate, has an invariant theory; but actually in comparison with that of the linear group, practically nothing is known about the others.

The equations (3), n in number, have n^2 coefficients, c_{ij}, among them. These, arranged in columns as suggested by each y, and rows as by each x, form a square array or matrix M. If $|c_{ij}|$ denote the determinant of this array, then, for both T and T^{-1} to exist, $|c_{ij}|$ must not be zero. If this is the only condition governing the coefficients c_{ij}, the corresponding group G of transformations is called the *general projective group* for n variables. The names binary, ternary and quaternary are used when $n=2, 3$ and 4 respectively.

Historically, this general group was studied without any serious reference to its subgroups, until comparatively recently. Roughly speaking, the second half of last century was the binary era, wherein most of the problems of binary forms were solved, so that the accounts given in the well-known treatises * are substantially complete. The present century represents the quaternary era, starting perhaps with a masterpiece by Gordan † on the invariant theory of two quadrics homogeneous in four variables, a work which successfully extended, into at first sight an elaborate and most unpromising field, symbolic methods originally designed for binary forms. Quite recently ‡ some progress has been made in the theory of n-ary forms, while the ternary case may be said to overlap both binary and quaternary periods.

In this treatise by Weitzenböck (incidentally reviewed in the *Gazette*, xii. pp. 122-124) will be found a full account of the invariant theory answering to subdivisions of the general linear group. Let a central conic C be fixed in a plane π. Answering to any pair of Cartesian axes Ox_1, Ox_2 intersecting at its centre, there will be a quadratic form

$$f \equiv ax_1^2 + 2hx_1x_2 + bx_2^2,$$

such that $f=1$ is the equation of the conic. The totality of all such frames of axes answers to the general binary linear group. If we impose the condition that the axes always should be rectangular, we obtain the orthogonal subgroup. If we further impose the condition that the only change of axes should be a twist through one or more right angles, we obtain a finite or modular subgroup. In these illustrations we have the key to the lines of research in the subgroups. The modular theory which dates from a memoir by Dickson § is receiving close attention in America. It is highly abstract, and one does not at all see whither it is leading. The orthogonal subgroup, however, and kindred subgroups which Weitzenböck considers, lead by pleasant bypaths back to the general group, and so do not provide anything essentially new.

This brings us to the consideration of what are called *ground forms*, which have been slurred over in the above, but no longer can be neglected. The conic, just noticed, has an equation with a set of coefficients a, h, b. This set is what Study calls the *kernel* of the form. Now when the same conic is referred to new axes at its centre, it takes on a new kernel a', h', b', say. We write

$$T: \quad x \rightarrow x',$$

$$(a, h, b \between x_1, x_2)^2 = (a', h', b' \between x_1', x_2')^2,$$

this last to represent the conic in its new dress. If this is an identity for all values of x, it implies that a', h', b' are linear functions of a, h, b. Hence

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* Elliott, *The Algebra of Quantics* (1895 and 1912); Grace and Young, *The Algebra of Invariants* (1903).

† *Math. Annalen*, 56 (1903).

‡ Cf. Weitzenböck, *Invariantentheorie* (Groningen, 1923), and *Trans. Camb. Phil. Soc.*, xxi. (1909), 197-240; *Proc. Edinburgh Roy. Soc.*, xlv. (1925), 149-165.

§ *Transactions American Math. Soc.*, 10 (1908).

the kernel undergoes a linear transformation, T_a say, because the variables x undergo T. Obviously this feature is also true of forms in any number of homogeneous variables and of any order. The transformation T_a is said to be *induced* by T. Given T, T_a is known. Can we reverse this ? This is the *equivalence problem*, which is too vast a theory to embark upon now : references can be found in Meyer. Setting this converse aside, we have the simultaneous change as indicated by

$$T: \quad x \to x',$$
$$T_a: \quad a \to a'$$

for transformation of variables x and coefficients a. If now ϕ is a function of these variables x and coefficients a, such that

$$\phi(x_1, x_2, \dots, a_1, a_2, \dots) = \phi(x_1', \dots, a_1', \dots),$$

then ϕ is an *absolute covariant* (or concomitant) of the *ground form* whose coefficients are $a_1, a_2 \dots$. Further, if ϕ satisfies the modified condition

$$\phi(x_1, \dots, a_1, \dots) = \lambda\phi(x_1', \dots, a_1', \dots),$$

λ being a function only of the coefficients c_{ij} of the transformation T, then ϕ is a *relative* concomitant. It can be proved that λ is necessarily a power of $|c_{ij}|$, the determinant of transformation. Owing to this and to the general idea of homogeneity running through the theory, relative concomitants are as important as absolute.

The central theorem of the invariant theory is the Gordan-Hilbert Finiteness Theorem. In 1868 Gordan proved that every rational integral concomitant of a given binary form was expressible rationally and integrally in terms of a finite number of concomitants. These last make what is called the irreducible system of the given ground form. This theorem was soon extended to any finite number of ground forms, and later by Hilbert to ternary and higher orders of variables. Gordan's proof carried with it the actual method of determining such a finite system for simpler cases, such as the binary quintic or sextic. Hilbert's more general proof gave no clue how any actual system could be found. Beyond binary forms very little is known of such systems for cubic and higher orders. But a great deal of work has been done in the last twenty years dealing with systems of linear and quadratic forms in three or four homogeneous variables, both for general and restricted transformations. Such work is mostly to be found in the Vienna *Sitzungsberichte*, the *Transactions of the American Mathematical Society* and the *Proceedings of the London Mathematical Society*.

There is much that may well prove of geometrical interest in what has been so accumulated in the algebra, which would furnish many a theme of interest for those who have a taste for methods of vectors or of projective geometry mixed with a little algebra. The geometrical treatment in the treatise by Grace and Young can readily and fruitfully be followed up for higher dimensions.

Besides touching on branches of mathematics already mentioned, the invariant theory on its formal side is closely connected with determinants and with the theory of finite groups. The finished product of one mathematical department is the raw material of another. For this reason, which is far from the only reason, the study of determinants and of finite groups for their own sake is important, because they provide the tools of the invariant theory. I may perhaps refer to writings by Sir Thomas Muir * and Professor E. T. Whittaker † on determinants to show a particular point of contact with invariants.‡ The need for more elaborate functions generalizing on deter-

* *Proc. Edinburgh Royal Society*, 44 (1923-4), 51-55.

† *Proc. Edinburgh Math. Soc.*, 36 (1917), 107-115.

‡ Cf. *Proc. London Math. Soc.*, 2. 22 (1923), 495-507.

minants is obvious if further advance in the general invariant theory is to be made. It is therefore of great interest to note how workers in determinant theory are coming to exactly the same conclusion. Major P. A. Macmahon, who has recently published a great work on determinants in the *Transactions of the Cambridge Philosophical Society*, talks of determinants as "still in their infancy."

More fundamental than determinants is the theory of finite groups and what Dr. A. Young has called Quantitative Substitutional Analysis. Such a theory is adumbrated in the last chapter of the *Algebra of Invariants*. There is a famous series of differential operators, called the Gordan-Capelli series, which holds a place in algebra of importance comparable to that of Taylor's Series in analysis. Now Young discards this series in favour of his own, which is based on purely *algebraic* operations consisting of substitutions; and his own is equally as effective in giving results as the Gordan-Capelli series. There are also grounds for hope that its effective use may be greatly enlarged. It is always risky to make prophecies as to lines of research, but the time is ripe for something to take place in algebra akin to the discovery of logarithms in arithmetic. The fundamental process of algebra is the permutation or rearrangement of a finite number of things. Now, compared with our knowledge of the way arithmetical processes combine, our knowledge of the behaviour of algebraic processes is still quite clumsy and tedious. Are any short cuts possible, and do they follow laws which are likely to be discovered? These are the really interesting questions arising to-day out of this branch of algebra.

H. W. TURNBULL.

GLEANINGS FAR AND NEAR.

392. Among the "three ways of distinguishing the sexes in the English language" are "change of termination, as prince, princess, phenomenon, phenomena."—*The Elements of English Grammar*, by John Dalton, 1801 [dedicated to Horne Tooke].

393. On the morning of his execution, 30 January, 1649, King Charles "likewise commanded Mr. Herbert to give his Son, the Duke of York, his large Ring Sun-Dial of Silver, a Jewel his Majesty much valu'd; it was invented and made by Mr. Delamaine, an able Mathematician, who projected it, and in a little printed Book shew'd its excellent Use, in resolving many Questions in Arithmetick, and other rare Operations to be wrought by it in the Mathematicks."—Herbert, *Memoirs of Charles I.*, 1815, p. 187.

394. Richard Delamain (fl. 1630) dedicated to King Charles his *Grammelogia, or, the Mathematical Ring . . . Shewing . . . how to resolve and work all ordinary operations of Arithmeticke*, London, 1630. See *D.N.B.* and Cajori, *William Oughtred*, 1916.

395. Then on week-days, by five o'clock in the morning, kindling the fire, setting on the porridge-pan, placing an old copy of Walkinghame's *Arithmetic*, or Pudey's *Murray*, pen, ink, and "posting-book" on the table, and in the light of a tallow candle (if winter time), with my foot on the trowler of a cradle, rocking a cross baby to quietness, whilst mother got a snatch of sleep, how I copied out rules of calculation, or learned to "parse." In comparison with such studies as these, how poor and pitiable are the strivings for a senior wranglership.—*Memoirs of Morgan Brierley* (1900), p. 8.

396. One of the most accomplished of English mathematicians, once a professor in the Royal College at Sandhurst, had not learned a letter of the alphabet when he was twenty years of age. From a little lad he had worked in a coal-pit.—*Memoirs of Morgan Brierley* (1900), p. 13.

THE COMBINATORY METHOD IN ANALYSIS SITUS.*

By M. H. A. Newman, M.A.

1. Analysis Situs, or Topology, is the investigation of those properties of surfaces and spaces which are unaltered by deformation, that is, by bending and stretching without tearing. For example, a circular disc with a hole in it can be stretched so that its boundary circuits have any of a great variety of forms, but the *number* of circuits (two) remains unaltered—it is a *topological invariant.* The fundamental problem of topology (at present unsolved) is to determine necessary and sufficient conditions that a given surface or space A should be deformable into a given surface or space B.

The first step in the investigation must of course be to assign precise meanings to the non-mathematical terms *space* and *deformation*, either constructively, by defining them in terms of more familiar entities, or axiomatically by stating the properties that characterise them.

A first manner of definition—perhaps the most natural for modern mathematicians—is to regard spaces as infinite aggregates of things called points, distinguished from other, structureless, infinite aggregates by the property that with every point P of a *space* are associated certain sets of points called the *neighbourhoods* of P. If then E is a set of points belonging to a space S, and P is any point of S, P is a *limit-point* of E if every neighbourhood of P contains a point of E distinct from P. The meaning of " limit point " being settled, it is easy to define " deformable into." The space A is *deformable into* the space B, or to use the more common expression, the spaces A and B are *homœomorphic*, if a (1, 1) continuous correspondence, μ, can be set up between the points of A and the points of B; *i.e.* a correspondence such that to a point P of A corresponds just one point, μP, of B and to Q of B corresponds just one point, $\mu^{-1}Q$, of A, in such a way that if X is a limit point of any set E contained in A, then μX is a limit point of μE, and, similarly, if Y of B is a limit point of the set F in B, then $\mu^{-1}Y$ is a limit point of $\mu^{-1}F$.

2. There is, however, a second, quite different way of interpreting the words " deformation " and " n-dimensional space," which is suggested to some extent by the (known) solution of the fundamental problem in two dimensions. If it is for a moment assumed that some meaning has been assigned to the expressions " surface," " piece," etc., the necessary and sufficient conditions that two surfaces, or pieces of surfaces, should be deformable one into the other may be stated as follows :

Let the two surfaces, A and B, be supposed divided into polygonal pieces. The edges of the polygons must not cross themselves, and two edges have in common either an end point of both or nothing at all. Clearly if the word " surface " has been satisfactorily defined every edge belongs either to one or to two polygons. (There is no implication that the edges are " straight," whatever that means.) The end points of the edges are called the *vertices* of the surfaces. If a_0 is the number of vertices of A, a_1 the number of edges, and a_2 the number of polygonal pieces, the number $a_1 - a_0 - a_2$ is called the *characteristic* of A. The necessary and sufficient conditions that A and B should be homœomorphic are :

(1) their characteristics must be equal ;
(2) the number of bounding circuits must be the same in both ;
(3) both must be 1-sided or both 2-sided.

The third condition will be explained in a moment.

Now the first of the three conditions is clearly not concerned at all with the constitution of the polygonal pieces and their edges as sets of points, but only

* Read at the *British Association* Meeting, Oxford, 1926.

with the way the pieces fit together—the co-incidences between their vertices and edges. In fact the following scheme,

$$X_2: \alpha\beta\theta,\ \beta\gamma\delta\theta,\ \delta\epsilon\zeta\eta\alpha\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{I.}$$

giving the vertices, in order, of each polygon of the piece of surface, X_2, represented in Fig. 1, contains all the information necessary for applying the

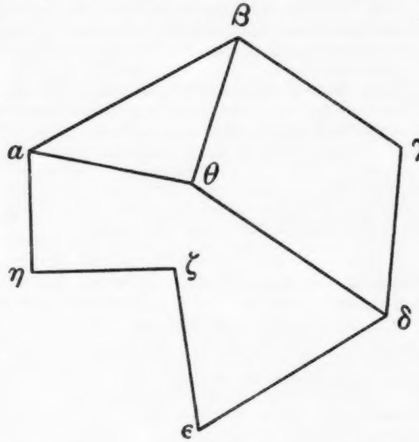

FIG. 1.

first test to X_2: *the characteristic number of X_2 is a property of the Scheme I. itself.* A little consideration shews that the number of bounding circuits of X_2 and its two-sidedness are also properties of the Scheme I. A *boundary edge* of a piece of surface is an edge belonging to only one of its constituent polygons; other edges are *internal.* From the Scheme I., without reference to Fig. 1, it is clear that the boundary edges of X_2 are $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\epsilon$, $\epsilon\zeta$, $\zeta\eta$, $\eta\alpha$, and that these form a single polygon. A surface or piece of surface is *two-sided* if a sense can be given to every constituent polygon in such a way that internal edges are described in opposite senses in the two polygons to which they belong. If this is not possible the surface is *one-sided.* It is clear, *from the Scheme I.*, that X_2 is two-sided. For if the polygons are given the senses $\alpha\beta\theta$, $\beta\gamma\delta\theta$, $\delta\epsilon\zeta\eta\alpha\theta$, the edge $\beta\theta$ has the sense $\beta\theta$ in $\alpha\beta\theta$ and $\theta\beta$ in $\beta\gamma\delta\theta$, and similarly $\delta\theta$ and $\alpha\theta$ are described once in either sense. On the other hand, the scheme

$$Y_2: \alpha\beta\gamma\delta,\ \alpha\beta\delta\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{II.}$$

represents a one-sided surface. For if the senses $\alpha\beta\gamma\delta$, $\alpha\beta\delta\gamma$ are adopted, $\alpha\beta$ has the same sense in both polygons, and so it has if both senses are reversed; while if only one is reversed $\gamma\delta$ has the same sense in both polygons. By inspection of the Scheme II. alone, without forming any mental image of the surface it represents, we can say that Y_2 is one-sided. Actually the surface (being obtained from the "rectangle"

$$\alpha\beta\gamma\delta,\ \alpha'\beta'\delta\gamma,$$

by letting α coincide with α' and β with β') can be realised by joining the ends of a paper strip so that diagonally opposite corners coincide. It is a familiar fact that any two points of such a figure can be joined without crossing the free edge of the paper.

3. It is established, then, that all three of the conditions (1), (2), and (3) on p. 222 really state relations which must hold between two arrangements, each of a finite number of letters. The *solution* of the fundamental problem in two dimensions having this form, it is natural to enquire whether the *problem itself* should not be stated so as to have reference to only a finite collection of things—whether there are not definitions of "2-dimensional space" and "deformable into" which are expressed in terms of schemes like I. and II., without introducing the notions of the Theory of Infinite Aggregates. Finally, if such definitions can be found in two dimensions, we may expect them to exist for all dimension numbers.

A theory of n-dimensional manifolds based on such a set of definitions was first systematically developed by *Dehn* and *Heegaard* in the article "Analysis Situs" in Bd. III. of the *Encyklopädie der Mathematischen Wissenschaften.* The crucial definition is, of course, that of "deformable into" or, as we shall call it in the combinatory theory, "topologically equivalent."

* Consider first the 1-dimensional case. All we need to know of a 1-dimensional set is its *vertices* and which pairs of vertices are the extremities of an *edge.* The set can, then, be specified by such a scheme as

$$\alpha\beta,\ \beta\gamma,\ \beta\delta,\ \delta\alpha. \qquad\text{III.}$$

An *internal transformation* of a 1-dimensional set is effected by choosing a new vertex, ξ, and replacing any edge, $\alpha\beta$, by the two edges $\alpha\xi$ and $\xi\beta$.

E.g. the 1-set, $\alpha\beta,\ \beta\xi,\ \xi\gamma,\ \beta\delta,\ \delta\alpha,$

is obtained from III. by an internal transformation. Two 1-dimensional sets are *topologically equivalent* (or, simply, *equivalent*) if there is a third 1-dimensional set which is obtainable from both by finite sequences of internal transformations.†

A 1-*dimensional element* is a 1-set equivalent to $\alpha\beta$;

a 1-*dimensional sphere* is a 1-set equivalent to the 1-set $\alpha\beta,\ \beta\gamma,\ \gamma\alpha$.

Clearly a 1-sphere is completely specified by giving the cyclical order of its vertices; the scheme of the 1-sphere with vertices $\alpha, \beta, \gamma, \ldots \delta$, in that order, may therefore be abbreviated to

$$\alpha\beta\gamma \ldots \delta\alpha.$$

Next consider the 2-dimensional case. A 2-dimensional set is a set of 1-dimensional spheres, called the *cells* of the set; and it is said to undergo an *internal transformation* when either (a) an edge $\alpha\beta$ is replaced, in every cell containing it, by the two edges $\alpha\xi\beta$, ξ being a "new vertex"; or (b) a cell $\alpha\beta \ldots \gamma \ldots \delta\alpha$ is replaced by the two polygons $\alpha\beta \ldots \gamma\xi \ldots \eta\alpha$ and $\alpha\eta \ldots \xi\gamma \ldots \delta\alpha$, where $\alpha\eta \ldots \xi\gamma$ is a new 1-element joining α to γ. (The new 1-element may be $\alpha\xi\gamma$ or simply $\alpha\gamma$ if there is not already such an edge in the figure.)‡

For example, from the 2-set,

$$\Delta_2 \colon \alpha\beta\gamma\alpha,\ \beta\gamma\delta\beta,\ \gamma\delta\alpha\gamma,\ \delta\alpha\beta\delta,$$

we may obtain $\alpha\xi\beta\gamma\alpha,\ \beta\gamma\delta\beta,\ \gamma\delta\alpha\gamma,\ \delta\alpha\xi\beta\delta$

by an internal transformation (i.t.) of type (a); or

$$\alpha\beta\gamma\alpha,\ \beta\gamma\delta\beta,\ \gamma\delta\alpha\gamma,\ \alpha\eta\beta\alpha,\ \alpha\eta\beta\delta\alpha$$

by an i.t. of type (b).

Two 2-dimensional sets are defined to be *topologically equivalent* if there is a third 2-dimensional set obtainable from both by internal transformations.

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* The next few paragraphs give the general idea of the Heegaard-Dehn system, but the details are different at a number of points.

† More accurately, the third 1-set must be *congruent* to sets obtainable by internal transformations from the other two. Two sets are congruent if they can be specified by the same scheme.

‡ These definitions should, of course, be "realised" with the help of a figure.

The *boundary* of a 2-dimensional set is the set of edges belonging to only one cell.

A 2-*element* is a 2-set equivalent to the 2-set with the single cell $\alpha\beta\alpha$.

The boundary of a 2-element is clearly a 1-sphere.

A 2-*sphere* is a 2-set equivalent to Δ_2 (above).

A 2-*manifold* is a 2-set such that the cells which contain any vertex form a 2-element.

It should now be plain how the definitions of n-dimensional equivalence, n-dimensional sphere, $(n+1)$-dimensional set are built up step by step. We give the 3-dimensional definitions:

A 3-*set* is a set of 2-spheres (the 3-cells of the set) with the following rules for *internal transformation*:

(1) an edge may be subdivided by the introduction of a new vertex;
(2) a polygon belonging to one more or 3-cells may be subdivided;
(3) a 3-cell, Σ_2, may be subdivided.

(1) and (2) have precisely the same meanings as in the 2-dimensional case. To effect an i.t. of the third kind a 1-sphere Π_1 lying in Σ_2 (*i.e.* formed out of its edges) is chosen, and a "new" 2-element, E_2, is taken, having Π_1 as its boundary but containing no other vertex or edge of the given 3-set. It can be proved that any 1-sphere lying in a 2-sphere divides the 2-sphere into two 2-elements, of both of which it is the boundary; and that two 2-elements having their boundaries, but nothing else, in common together form a 2-sphere. Let E_2^1 and E_2^2 be the 2-elements into which Π_1 divides the 2-sphere bounding Σ_2. It is an *internal transformation of the third kind* to replace the cell Σ_2 by the two cells whose boundaries are $E_2+E_2^1$ and $E_2+E_2^2$.

4. The Heegaard-Dehn definitions, of which some account has now been given, demonstrate the logical possibility of the combinatory method, but it has been found rather difficult actually to find proofs of the fundamental theorems. The crucial part of such proofs is usually the consideration of the effect of changing the order of two or more internal transformations, and the figures which arise are so complicated that they themselves require powerful general theorems for their consideration. This leads to a great interlocking of inductive hypotheses, not only in the definitions but also in the proofs.

Now the characteristic property of combinatory definitions is that the "equivalence" of two sets should be referred to the possibility of transforming one set into the other by a finite series of "moves" of certain prescribed types. It is not essential that the prescribed moves should be the "internal transformations" of Heegaard and Dehn; and there seem in fact to be advantages in choosing as fundamental moves not the *division* of cells but the *addition* or *removal* of a cell. This allows the use of *simplexes* as building units and avoids the obstructive generality of Heegaard-Dehn transformation.

Let us make a final fresh start.

An n-*array* is formed from a finite set of objects by choosing as the units of the array certain of the groups of $n+1$ objects contained in the set. The objects are called the *vertices* of the n-array, and if $0 \leqslant k \leqslant n$ any $k+1$ vertices all belonging to the same unit form a k-*component* of the array. An $(n-1)$-component is a *face*.

(For example the objects $\alpha, \beta, \gamma, \delta, \epsilon$ are organised into a 3-array by choosing $\alpha\beta\gamma\delta$, $\alpha\gamma\delta\epsilon$, and $\beta\gamma\delta\epsilon$ as the units of the array. $\beta\gamma$ is then a 1-component and $\gamma\delta\epsilon$ a face; $\alpha\beta\epsilon$ is not a component (although all its vertices belong to the array), because its vertices are not contained in any one unit. A *different* array is obtained if other units—say $\alpha\beta\gamma\delta$ and $\alpha\beta\delta\epsilon$—are chosen. The units of a 2-array are of course to be thought of as triangles and those of a 3-array as tetrahedra.)

An n-array Γ *contains* a k-array Θ only if every unit of Θ is a unit or component of Γ.

An *n-simplex* is an n-array with only one unit.

If Γ, Δ, ... are n-arrays, $\Gamma+\Delta+\ldots$ is the n-array whose units are the units of all the arrays. If S and T are m- and n-simplexes with no common vertex, ST is the $(m+n+1)$-simplex containing all the vertices of both. If Γ and Δ are arrays having no vertex in common, $\Gamma\Delta$ is the sum of all the products ST, where S is a unit of Γ and T a unit of Δ.

The sum of the faces of an n-array, Γ, that are contained each in only one unit is the *boundary* of Γ, written $\overline{\Gamma}$. A component not in the boundary is *internal.*

It is easy to prove that *if* Γ *and* Δ *have no common vertex* $\overline{\Gamma\Delta}$ *is* $\Gamma\,.\,\overline{\Delta}+\Delta\,.\,\overline{\Gamma}$. (Here $\Gamma\,.\,\overline{\Delta}$ must be omitted if Δ is unbounded, and replaced by Γ if Δ is a vertex. The examples of the theorem in which (a) Γ and Δ are 1-simplexes, and (b) Γ is a vertex and Δ a 2-simplex, should be considered with figures.)

These notations having been agreed upon, we can proceed to the definition of a "regular move."

If Γ is a bounded n-array and S an n-simplex not belonging to Γ, S is said to have *regular contact* with Γ if it is the product of two components, U and V, such that

A (i). U belongs to $\overline{\Gamma}$.

A (ii). U is interior to $\Gamma+UV$.

A (iii). V does not belong to Γ.

(The effect of this definition is that the contact is regular if the components of S belonging to Γ lie in $\overline{\Gamma}$ and form a connected $(n-1)$-dimensional piece of $\overline{S}$, but not the whole of it.)

It is a *move of type* 1 to add a simplex to an array (of the same dimension number) with which it has regular contact; it is a *move of type* 2 to remove a simplex from an array with the rest of which it has regular contact.

If Γ is an n-array containing the array $X\,.\,\overline{Y}$ (where X and Y are simplexes and the dimension number of $X\,.\,\overline{Y}$ is n), it is a *move of type* 3 to replace $X\,.\,\overline{Y}$ by $Y\,.\,\overline{X}$, provided neither X nor Y belongs to $\Gamma-X\,.\,\overline{Y}$.

(The arrays $X\,.\,\overline{Y}$ and $Y\,.\,\overline{X}$ together make up $\overline{XY}$, the boundary of the $(n+1)$-simplex XY. It is clear that if a move of type 1 or 2 is performed on a bounded n-array, a move of type 3 is performed on its boundary.)

If the *bounded* array Γ can be transformed by a finite succession of moves, of any or all the types 1, 2, and 3, into the array Δ,* Γ is said to be *topologically equivalent* (or simply *equivalent*) to Δ, and we write: $\Gamma\rightarrow\Delta$. If only moves of types p and q are required we write $\Gamma\underset{pq}{\rightarrow}\Delta$.

If in the unbounded arrays Γ and Δ units S and T exist such that the bounded arrays $\Gamma-S$ and $\Delta-T$ are equivalent, then Γ and Δ are, by definition, equivalent. It may be shewn that if Γ (unbounded) can be transformed into Δ by moves of type 3 only, then $\Gamma\rightarrow\Delta$, and we write $\Gamma\underset{3}{\rightarrow}\Delta$.

We can now define an n-element to be an n-array equivalent to an n-simplex; and an n-sphere to be an n-array equivalent to the boundary of an $(n+1)$-simplex. An *n-manifold* is an n-array such that the sum of the units containing any vertex has the form $a\Pi_{n-1}$, where a is a vertex and Π_{n-1} is either an $(n-1)$-sphere or an $(n-1)$-element. It may then be shewn that elements and spheres are manifolds, and that the sum of the units of an n-manifold containing a k-component S_k has the form $S_k\Pi_{n-k-1}$, where Π_{n-k-1} is an $(n-k-1)$-element or $(n-k-1)$-sphere.

5. From these definitions the theory can be built up without any very considerable use of inductive reasoning from one dimension number to another. The simplicity of the figures directly affected by a single move allows many lemmas to be proved by an almost purely algebraical use of known results,

* More accurately into an array congruent to Δ. Cf. footnote, p. 224.

without the necessity of considering their "meanings." As a specimen, consider this lemma, which has important applications:

Let P, Q, R, be simplexes of which no two have a common vertex, and let Γ be an array having the same dimension number as $P . \overline{Q} . \overline{R}$. Then

$$\Gamma + P . \overline{Q} . \overline{R} \underset{3}{\rightarrow} \Gamma + Q . \overline{P} . \overline{R},$$

provided Γ contains neither P nor Q.

The proof is as follows:

If R is a vertex the theorem becomes (in virtue of a convention already explained)

$$\Gamma + P . \overline{Q} \underset{3}{\rightarrow} \Gamma + Q . \overline{P},$$

which is true by the definition of a move of type 3. Suppose then that it is true when R is replaced by a simplex of lower dimension number. Let ξ be a vertex of R and R' the opposite face, so that R is $R'\xi$.

$$\begin{aligned} \Gamma + P . \overline{Q} . \overline{R} &\text{ is } \Gamma + P . \overline{Q} . R' + P . \overline{Q} . \overline{R'} . \xi, \\ &\underset{3}{\rightarrow} \Gamma + P . \overline{Q} . R' + \overline{P\xi} . Q . \overline{R'} \text{ (by the inductive hypothesis)}, \\ &\text{ is } \Gamma + P . \overline{Q} . R' + Q . P . \overline{R'} + Q . \overline{P} . \xi . \overline{R'}, \\ &\text{ is } \Gamma + P . \overline{QR'} + Q . \overline{P} . \xi . \overline{R'}, \\ &\underset{3}{\rightarrow} \Gamma + \overline{P} . QR' + Q . \overline{P} . \xi . \overline{R'}, \\ &\text{ is } \Gamma + Q . \overline{P} . \overline{R}. \end{aligned}$$

The narrowness of the range of admissible "moves" is of course in one way, initially, a disadvantage; for the equivalence of many pairs of arrays that follows immediately from the Heegaard and Dehn definitions must now be laboriously proved. A first stage in the development of the theory is reached when it is proved that all "Heegaard-Dehn" subdivisions of the cells of a manifold (and, in fact, all modifications of a much more general kind), do, in fact, lead to equivalent manifolds, in the sense that has just been defined. The new theory has, then, all the "mobility" of the old, but we have the additional result that a general subdivision can always be effected by a sequence of moves of the three standard types. How this result is proved cannot even be indicated here, but it may be mentioned that it is found that the three moves are not all independent; a move of any type applied to a manifold can be effected by moves of the other two types—*i.e.*, if $\Lambda \rightarrow M$, then $\Lambda \underset{13}{\rightarrow} M$, $\Lambda \underset{23}{\rightarrow} M$, and $\Lambda \underset{12}{\rightarrow} M$. (The reader will easily convince himself that this is so in the 2-dimensional case by experimenting with figures.)

References. The article of *Dehn* and *Heegaard* is

Encyk. der Math. Wiss., III AB 3, "Analysis Situs."

See also *E. Bilz*, Math. Zschr. 18 (1923), 1.

The theory outlined in the last two pages is developed in two papers:

M. H. A. Newman, Foundations of Combinatory Analysis Situs, I. and II., Proc. Roy. Ac., Amsterdam, 29 (1926), pp. 611-626 and 627-641.

Alternative theories have also been given by

H. Weyl, Análisis Situs Combinatorio, Revista Matém. Hispano-Americano, 1923, and

H. Kneser, Topologie der Mannigfaltigkeiten, Jahresber. der Deutschen Math.-Ver., 1925.

To avoid the difficulties that beset a constructive definition of *spheres* (in the way that has here been adopted) Weyl suggests defining them axiomatically, as things having the properties required of them to play satisfactorily the part of "cells" in a manifold. This, of course, leaves open the question whether two objects satisfying the axioms can always be obtained from one another by some train of "moves."

M. H. A. NEWMAN.

TO DRAW A CAYLEY CONFIGURATION.*

By Prof. Norman Anning, M.A.

In the preface to his *Principles of Geometry*, Professor Baker states that the "Foundations" volume "assumes only those relations of position, for points, lines and planes, which, furnished with a pencil, some rods and some string, a student may learn by drawing diagrams and making models." When the beginner attempts to make these diagrams, it is not unusual for one or more points to go off the page. To illustrate how, in an important set of cases, it is possible to get all of the diagram on the page at the first trial is the purpose of this note.

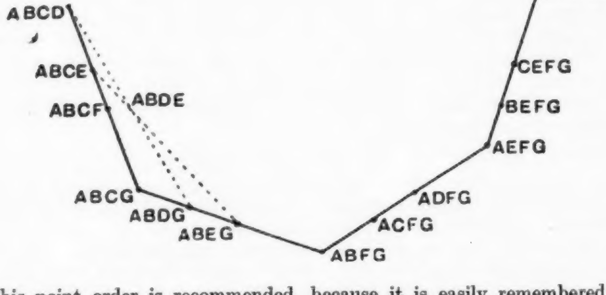
A Cayley configuration is one of $\binom{n}{r}$ points and $\binom{n}{r+1}$ lines, with $(r+1)$ points on every line and $(n-r)$ lines through every point. Such a configuration is native in space of $(n-r)$ dimensions, but can, by a sufficient number of successive projections, be reduced to a plane diagram.

Consider, for example, the self-dual configuration of $\binom{7}{3}$ points and $\binom{7}{4}$ lines. A convenient description of this four-space configuration is suggested by the symbol, 35—35—21—7—1, which is to be read "35 points, 35 lines, 21 planes, 7 three-spaces, 1 four-space." A more complete description is given by the fragment of Pascal's Triangle in the margin, which is to be read "four points on every line, ten points on five lines on every plane, etc."

1				
4	1			
10	5	1		
20	15	6	1	
35	35	21	7	1

Let us denote by A, B, C, D, E, F, G the 7 three-spaces, by $AB, AC, \ldots$ the 21 planes, by $ABC, \ldots$ the 35 lines, and by $ABCD, \ldots$ the 35 points, it being understood that AB is that plane which is common to the three-spaces A and B, etc.

Because the configuration belongs to four-space, draw four line segments and choose on them 13 points, which are labelled as in the diagram.



This point order is recommended, because it is easily remembered and extended or contracted to fit other cases. The obtuse angles, or at any rate the absence of straight or reflex angles on the inside of the frame, also contribute to the success of the venture. The remaining 22 points will be found as the construction proceeds. Thus the line ABD can be drawn because we

* In his article on "Konfigurationen" in the *Encyklopädie* Steinitz assigns the name "Cayley" to configurations of the type mentioned, because Cayley published a discussion of them in *Crelle* (1845).

have the points *ABCD* and *ABDG*. Similarly the line *ABE* can be drawn. These two lines lie in the plane *AB*, and by their intersection determine the point *ABDE*, and so on. If the points are labelled as soon as they are determined, it will be found that they fall into line in a truly remarkable fashion. (Two words of caution: while the lines *ABD*, *ACF* belong to the same three-space, *A*, they need have no point in common, and so no label exists for the point at which they may cross in the figure; with many lines in a restricted space some line may appear in the drawing to pass through a point which does not belong to it—the writer cannot suggest a way of avoiding such an accident.)

It is probable that no picture has ever been made of the self-dual five-space configuration, which has 126 points and 126 lines, 5 points on each line, and 5 lines through each point. Any student who wishes to profit by the above suggestions and attempt it should proceed as follows. Choose the five lines *ABCD, ABCI, ABHI, AGHI, FGHI*, and mark on them in order from end to end the 21 points *ABCDE, ABCDF, ABCDG, ABCDH, ABCDI, ABCEI, ABCFI, ..., EFGHI*. The remaining 121 lines and 105 points can then be constructed.

After the student has used this method for a few plane diagrams, it will not be hard for him to extend it to models of "rods and string."

University of Michigan, NORMAN ANNING.
Aug. 4, 1926.

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397. I am reminded of my old master in the Mathematics at Edinburgh, Professor Kelland (of whose "kindly spectacle" Mr. Stevenson has written so charmingly). When Kelland sat in the seat of judgment ... to his fellow-examiner he would say, touching the paper gently with his fingers, as though he would feel the beating heart that waited anxiously outside for the verdict: "We'll let the laddie through this time; he's done his best. *It's true his best is not very good!*"—S. R. Crockett, Dedication to *Lad's Love;* and in "The Biography of an 'Inefficient,'" *Bog-Myrtle and Peat.*

398. It was only necessary to look at the respective heads of the pupils to conclude that these young persons were engaged in mathematical problems, for there is nothing so discomposing to the hair as arithmetic. Mademoiselle Lange herself seemed no more capable of steering a course through a double equation than her pupils, for she was young and pretty, with laughing lips and fair hair, now somewhat ruffled by her calculations.—H. Seton Merriman, *The Isle of Unrest*, c. v.

399. The pious and gentle Rev. John Buckley, of Friarmere ... scholar and gentleman, saint and patriarch, village schoolmaster, a Church of England clergyman, good preacher and diligent pastor ... "passing rich on forty pounds a year." He was a good geometer, and contributed to the mathematical periodicals of his day in company with Wolfenden, Hilton (editor of the *Liverpool Student*), Kay, and Butterworth. ...—*Memoirs of Morgan Brierley* (1900), p. 13.

400. It may be said that Bradley changed the face of astronomy. ... Nevertheless the name of Bradley hardly appears in popular works, nor will do so until the state of astronomy is better understood. Let any man set up for the founder of a sect, and begin that by asserting that he has found out the cause of attraction, or the structure of the moon; let him exalt himself in the daily papers, and he must be unfortunate indeed if in less than three years he is not more widely known in this country than its own Bradley, one of the first astronomers of any.—De Morgan, *Penny Cyclopaedia.*

THE BRITISH ASSOCIATION MEETING, OXFORD, 1926.

By W. J. Langford, B.Sc.

The ninety-sixth meeting of the British Association was one which will doubtless live long in the memories of all those present. The Association was received in Oxford in the characteristic way that only the older University towns are able to pursue. The cooperation between the Academic and the Civil Authorities makes it possible to meet the often unexpressed, but nevertheless conscious wishes of all members.

The Presidency of H.R.H. the Prince of Wales was at once an honour to the Association, and a further concrete example of the interest taken in the advancement of learning by all members of the Royal Family. In his address the President gave an admirable précis of the history of the Association—a précis which must have been of great value to all who heard it, enabling them to form a true conception of the rapidity with which Science has progressed within the last century.

The Oxford meeting of 1860 had witnessed the memorable battle between Wilberforce, Bishop of Oxford, on the one hand, and Huxley and Hooker on the other. Again, in 1894 Maxim had put before the Association his conviction of the possibilities of flight, while at the same meeting the first public demonstration of wireless signalling was given by Sir Oliver Lodge. Oxford thus seems to have been associated with many of the epoch-making contributions to Science which have been made public before the British Association.

While this short report in no way presumes to differentiate between papers read at the meeting, there is one fact, of an entirely different nature, which is worth further consideration. For the first time in the history of the Association a division was made in Section A. Though this partial segregation may have been of little moment to the physicists, there seems little doubt that it was welcomed by mathematicians, especially, it may be said, by the younger members whose study of the many branches of mathematics had left them little time to devote to the sister science. The duration of the meeting is too short to allow papers in physics and mathematics to be given to a joint meeting. Judging by the programme of last year it was the latter which failed to find its place. It is impossible to believe that the combined efforts of present-day mathematicians could have been portrayed in the two or three short papers which were read at that meeting. It may be that the formation of a specific branch has acted as a stimulus. The fact remains that Oxford was able to record a return to the expected standard of contribution. One is led to hope that the division may, if possible, become even more sharply defined in the future. This is in no way a reflection on the interest which papers in the realm of physics hold for mathematicians, but rather a plea that the future will see an increase in the presentation of accounts of original work which it is felt must exist.

The subject matter of the papers presented this year spreads over the whole field of mathematics. Mr. Ramsey's paper, reported in a previous issue of the *Gazette*, established a close connection with the super-theory of logic and philosophy, while a definite link with physics was provided by Prof. Milne's discourse on Maxwell's Law and Radiation. Prof. Volterra's interesting contribution on the apparent cyclic law which governs the maintenance of almost all species of fish, even when account is taken of the havoc wrought by those which prey on their lesser brethren, introduced a new conception in the "coefficient of voracity," and, further, provided an unusual link between zoology and mathematics.

Dr. Cherry enlarged upon the theory of orbital dynamics in considering the

solution of dynamical equations in the Hamiltonian form. From equations of the form

$$\frac{dx_r}{dt}=X_r(x_1x_2x_3\ldots x_n),$$

as a starting point, power series were derived by means of successive approximation, and after the introduction of a linear transformation of the type

$$H_2=\lambda_1x_1y_1+\ldots+\lambda_nx_ny_n,$$

the solution was exhibited in the form

$$x\,.\,y=P(a_re^{\lambda_rt},\beta_re^{-\lambda_rt},t).$$

The appearance of the variable t led to the consideration of secular terms, and the author then proceeded to examine the conditions under which these might be eliminated.

The papers in pure mathematics were too varied for a detailed account to be attempted here. A most interesting series of papers was given on integration and trigonometrical series, one of which was reported fully in the last number of the *Gazette*. Prof. Hardy dealt with series of the Fourier type, and Prof. Carathéodory put before the meeting two pieces of analysis, one of which was concerned with a periodic law in relation to a moving point. After hearing the admirable address by Prof. Fowler, the President of Section A, on the analysis of line spectra, the sub-section assembled to receive some short papers by distinguished visitors. Dr. Ostrowski brought forward a general theorem on the zeros of functions connected by a linear relation, which he proved after first considering a special case. Prof. van Dyck showed some excellent graphical work on algebraic operations with generalised quaternion expressions of the form

$$a_0\epsilon_0+a_1\epsilon_1+\ldots+a_n\epsilon_n.$$

The report of the Seismological Committee was characterised by a notable addition to the theory of the structure of the earth. Dr. Jeffreys explained briefly the reasons which had led him to assume that the central core of the earth was not of a metallic nature, but was actually a true fluid. It had been observed that a distortional wave incident on this core from above gave rise to reflected compressional and distortional waves, while the only waves to be transmitted were P waves. The failure to transmit S waves was most naturally explained by the assumption that the central core was a fluid, and recent data have almost all tended to confirm this theory.

There is no doubt that the meeting was once again a great success, and that the gathering together of the foremost scientific thinkers of the world serves a tremendous purpose in promoting informal, as well as formal, discussion on the many scientific problems of to-day. It is to be hoped that history will deign to repeat itself in such a way that from the papers contributed some far-reaching development will arise and this meeting be placed on an equality with its renowned predecessors.

Bec School, S.W. W. J. LANGFORD.

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401. It is said that the Marquis de l'Hôpital at the age of fifteen, happening to be in company with a number of savants at the house of the Duke de Roannez, when great admiration was expressed of a solution which Pascal had recently given of a problem relative to the cycloid, expressed his belief that the question was not beyond his own powers, and two days afterwards he supported his pretensions by answering it on different lines.

402. "Euclid ?" replied the flapper to her companion. "Yes, I've heard of him. Didn't he write 'The Eternal Triangle' or something ?"—*Manchester Guardian.*

THE ASYMPTOTES OF PLANE CURVES.

By H. G. Green, M.A.

While Professor Nunn's treatment of Asymptotes (*Math. Gazette*, May 1926) gives the elementary student a method for finding the equations of linear asymptotes, the present writer feels that there is some weakness in the method, as the geometrical consequences of the various types of asymptote on the general form of a curve are not emphasised. There is, moreover, a serious difficulty for the advanced student. Professor Nunn regards two parallel asymptotes as a degenerate case of the asymptotic parabola, thus throwing over entirely the ideas of multiple points at "infinity". This must lead to considerable modifications in the values and properties of Plücker's numbers. In fact, the change would affect the whole theory of the deficiency of curves.

The idea of the continuity of form as applied from curve to curve, when the equations contain similar terms with different numerical coefficients, is a dangerous one. For example, to leave the subject of asymptotes for an instant, many of the older text-books, without any actual statement, rather give the student the conception of a cusp as an extreme case of a node, whereas closer inspection of the branches shows that they are fundamentally different forms.

In the present article we venture to put forward a geometrical introduction to the study of asymptotes in the hope of removing these difficulties. With Professor Nunn we take as the definition of an asymptote: "An asymptote of a curve of the nth degree is the limit of a continuous series of parallel lines which cut the curve in less than n points and whose intersections therewith, as the lines approach the limit, become and remain farther from the origin than any given distance, however great."

To introduce the subject we use the processes of conical projection. With V as vertex we project a curve C (degree n) in the plane π into the curve C' in the plane π', AB and $A'B$ being the vanishing lines as shown in Fig. 1. With the exception of points of intersection with AB all finite points of C become finite points of C', and any line cutting C in n finite points becomes a line cutting C' in n finite points except when one of the intersections with C is also on AB. For the purposes under discussion it is not necessary to distinguish between real and imaginary points when counting the number of intersections of a line with the curve. If the student finds difficulty in following the projection from a figure it will be convenient to make use of a rudimentary model (an exercise book partly opened is excellent for the planes with some fixed point for V).

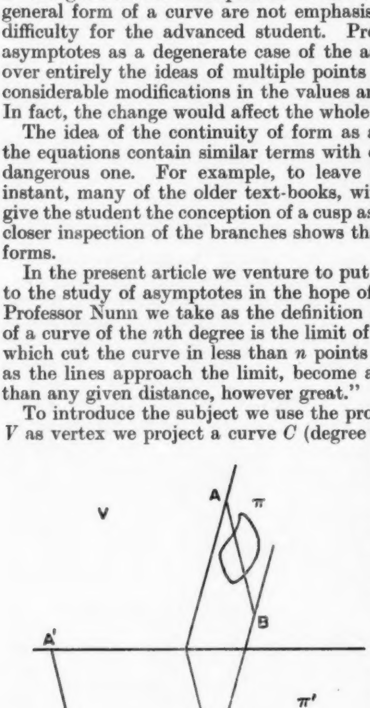

Fig. 1.

Let the point P be a simple point of intersection of C with AB; the pencil of straight lines through P is such that, as we approach the limiting tangent position, an intersection Q with the curve other than P tends towards P. Draw the tangent and letter the areas as in Fig. 2: projecting into the π' plane we obtain the corresponding part of the curve C' (Fig. 3). The pencil of lines through P becomes a set of parallel straight lines with a limiting

position corresponding to the tangent in the old figure, and as a line of the set moves towards this position the point of intersection Q' moves to greater distances from any fixed point in the plane, and finally disappears in the limiting position. The identity of the asymptote with the line thus obtained

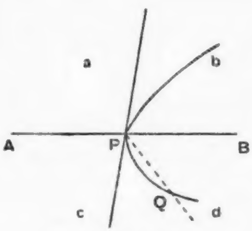

FIG. 2.

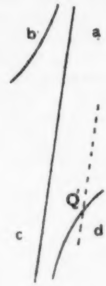

FIG. 3.

becomes established, and from consideration of the areas the general behaviour of the curve with regard to it is also obtained. Though the asymptote has been derived from a tangent in the π plane it must not, however, be assumed that it is a tangent to the C' curve: all we can say on this matter is that in the neighbourhood of the asymptote at great but finite distances there is

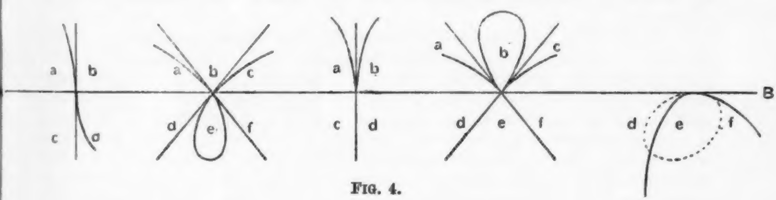

FIG. 4.

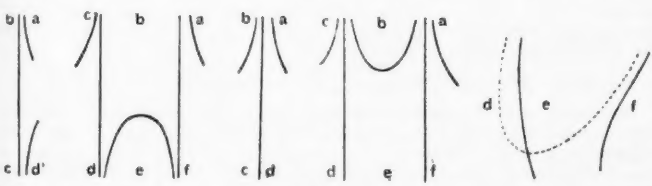

FIG. 5.

a section of the curve behaving as if the asymptote were a tangent. Proceeding we may take some of the simpler standard cases as in the first four curves of Fig. 4, deriving the types of asymptote shown in Fig. 5: others may be readily constructed in the same way.

When C is tangential to AB the construction breaks down. If the contact be simple we can draw the conic of closest contact with the C curve at this point, giving rise on projection to the "asymptotic parabola" (with not more than $2n-5$ finite intersections with the C' curve) as the corresponding approximation. More complicated types of contact give rise, in a similar way, to

asymptotic curves of higher orders. The simplest case is shown in curve 5 of Fig. 4.

Again, we do not infer that the singularities of C along AB persist in C', but we do infer that at great but finite distances along the asymptotes (or asymptotic curves) sections of C' are behaving as if there were singularities; and, if for this purpose we assume their existence, we can apply the Plücker relations without modification. With this understanding we speak of an asymptote as being simple, inflexional, with four-point contact or cuspidal, etc., as the case may be.

To complete the geometrical problem we must start with a given curve C' and examine the possibility and behaviour of a series of parallel straight lines, of which, in general, each has $n-k$ finite intersections with C'. We arrange the projection back into a π plane in such a way that the vanishing line $A'B'$ cuts C' in n different points: from the definition no asymptote is in a direction parallel to $A'B'$. In the C figure we obtain from the series of parallel straight lines a pencil with vertex on AB, of which any general lines (except n special ones derived from lines through the intersections of $A'B'$ and C') has $n-k$ points on C obtained by the projection. It follows that for the existence of the series, the vertex must be a k-ple point of C. But, taking any line through such a k-ple point, at least one other point of intersection tends towards it when the line tends towards a tangent position, and thus there is for C' a limiting position of the series, with all the conditions for a linear asymptote, corresponding to every tangent of the k-ple point which is not coincident with AB. We have in fact established that, if we can find a series of parallel lines in which any one has less than n intersections with the curve, and if we can determine a limiting position in which further reduction in the number of intersections occurs, we have completely satisfied the conditions for an asymptote.

It remains now to change the geometric results into an algebraic form. We take $u_n+u_{n-1}+u_{n-2}+\ldots+u_0=0$ as the equation of C', where u_{n-s} is a homogeneous function of degree $n-s$ in the variables x, y. To find the intersections of the curve with a line $lx+my=p$ we eliminate one of the variables and obtain an equation in the other of degree n in general. If, however, $lx+my$ be a factor of u_n the degree becomes $n-1$ at most, and an asymptote will be obtained for every value of p which still further reduces the degree. For brevity we indicate the procedure in a general form: the various elementary types may be investigated separately in the same manner, and the argument for two cases is shown in detail in the numerical example given at the end. Suppose that a factor $lx+my$ appear r times in u_n and t times in u_{n-s}, so that the equation may be put in the form

$$(lx+my)^r v_{n-r}+(lx+my)^{t_1} v_{n-1-t_1}+\ldots(lx+my)^t v_{n-s-t}+\ldots u_0=0.$$

In the process of the solution for intersections, the highest degrees obtained from u_n and u_{n-s} are $n-r$, $n-s-t$ respectively, and have as coefficients expressions in p of degrees r and t respectively. We now gather together the terms of highest degree, $n-k$ $(k \leqslant r)$, and, equating the sum of its coefficients to zero, we obtain the required p equation. Any root of this equation leads to a line of the set, $lx+my=p$, which has a further reduction of its number of intersections, *i.e.* an asymptote. In the equivalent figure we shall have a multiple point of order k having r points of intersection with AB, and comparing Figs. 2 and 3, 4 and 5, we see that a non-repeated root of the p equation corresponds to a single branch, a root occurring twice to a cuspidal branch, and so on. If on giving p its numerical value, the next term automatically vanishes and the one after does not, we have an inflexional branch (Figs. 4, 5): further reductions imply correspondingly higher orders of contact. If $k=r$ (which is always the case when the factor $lx+my$ occurs only once in u_n, $r=1$), then for any of the terms of highest degree after substitution $r=s+t$

and $r > t$, or the equation in p is of degree r. If $k < r$ (the term from u_n is not the highest), then for the terms of highest degree after substitution $k=s+t$; the greatest t is certainly less than k, and the number of p roots leading to linear asymptotes is less than the number of branches of the k-ple point. We conclude that the extra branches of C touch AB with contact which may be simple or may be of higher order, the numbers of branches and of points of intersection involved being $k-t$, $r-t$ respectively, where t has this greatest value, and for these sections in the C' curve approximations can only be obtained by means of curvilinear asymptotes.

Except in specially favourable cases the closer investigation of the bearing of the branch of a curve to its asymptote is beyond the range under discussion. In a complicated case it demands the use of series for which sound justification is difficult, though some degree of probability can be obtained by repeated approximation. Repeated approximations may also be made use of in the calculation of the equation of an asymptotic parabola, but the simplest method in practice is to assume that the series $y=mx+ax^{\frac{1}{2}}+b+c/x^{\frac{1}{2}}+d/x+\ldots$ represents at great distances this part of the curve, where $y-mx$ is the factor of u_n concerned and a, b, c, d, ... are unknown numbers. These numbers are found by the process of substitution in the equation of the curve and comparison of coefficients, the parabola required being $(y-mx-b)^2=a^2x+2ac$. Curvilinear asymptotes of higher order may be similarly investigated by means of series.

We take as a numerical example the curve

$$(x-y)^2(x+y)^3+4(x+y)(x-y)(x^2+y^2)-8(x^3+2y^3)-12x^2+y=0.$$

In the right-hand column we give the numerical values of the symbols used in the general discussion, showing the powers of the various terms in the form $\{n-s-t,\ t\}$.

(1) Possible asymptote $x=y+p$: substituting for x,	$\{3,2\}+\{3,1\}+\{3,0\}+\ldots$
$p^2(2y+p)^3+4(2y+p)(p)(2y^2+2yp+p^2)$	
$-8(3y^3+3y^2p+3yp^2+p^3)-12(y^2+2yp+p^2)+y=0.$	$k=2=r$.
Coefficient of y^3: $8p^2+16p-24$, $=0$ if $p=1$ or -3.	Of degree 2, from $\{3,2\}$, $\{3,1\}$, $\{3,0\}$.
,, ,, y^2: $12p^3+24p^2-24p-12$, $=0$ if $p=1$, $\neq 0$ if $p=-3$.	
,, ,, y: $6p^4+16p^3-24p^2-24p+1$, $\neq 0$ if $p=1$.	
Asymptotes: $x=y-3$, simple. $x=y+1$, inflexional.	

The curve behaves as if there were a double point at "infinity" in the direction $x=y$, with one inflexional branch.

(2) Possible asymptote $x=-y+p$: substituting for x,	$\{2,3\}+\{3,1\}+\{3,0\}+\ldots\ldots$
$(-2y+p)^2p^3+4(p)(-2y+p)(2y^2-2yp+p^2)$	
$-8(y^3+3y^2p-3yp^2+p^3)-12(y^2-2yp+p^2)+y=0.$	$k=2<3$.
Coefficient of y^3: $-16p-8$, $=0$ if $p=-\frac{1}{2}$.	of degree 1, from $\{3,1\}$, $\{3,0\}$.
,, ,, y^2: $4p^3+24p^2-24p-12$, $\neq 0$ if $p=-\frac{1}{2}$.	
Asymptote: $x=-y-\frac{1}{2}$, simple.	

The curve behaves as if there were a double point at "infinity" in the direction $x=-y$, with one of the branches touching the "line at infinity" in simple contact $(3-1=2)$. For this branch we can obtain an asymptotic parabola. A substitution $y=-x+\alpha x^{\frac{1}{2}}+\beta+\gamma/x^{\frac{1}{2}}+\ldots$ yields, on comparing coefficients after rather heavy but straightforward work, the series of values (1) $\alpha=0$, $\beta=-\frac{1}{2}$, $\gamma=0$—the linear asymptote again; (2) $\alpha^2=-4$, $\beta=\frac{5}{4}$, $\alpha\gamma=\frac{229}{32}$, and we have for the asymptotic parabola

$$(x+y-\tfrac{5}{4})^2=\tfrac{229}{16}-4x.$$

H. G. GREEN.

MODERN EXAMINATION TENDENCY.

By R. S. Williamson, M.A.

Recent examination developments have been influenced largely by two factors.

(1) The institution of Local Education Authorities in 1902. These Authorities have been able to survey over a wide range of Secondary Schools the working of the examination system and to estimate to some extent its effects on Secondary Education as a whole. Consequent on this, but not wholly the result of it, came some general reforms, *e.g.* the formation of the Secondary Schools Examination Council. But L.E.A.'s are also involved in the examination system as examiners, since they test candidates for scholarships and free places in Secondary Schools. Their examination view-point is therefore a wide one, embracing that of both teacher and examiner. The full effect of this dual point of view has not yet matured, but some innovations or improvements in junior scholarship examinations have already resulted from it ; for instance, the development of the oral examination and the use of the teacher's assessment of a pupil. In some areas, too, teachers participate in marking scripts. Examinations of School Certificate standard are so far little affected, but it would be in accord with the trend of events if the wide powers of University Examining Authorities were further modified and the school record of the candidate made to play a more important part in the examination than at present.

(2) Psychologists, working on their own particular problem—the study of intelligence—have entered the examination field and have challenged existing examination custom in two important aspects. Firstly, they have criticised, as unscientific, the rough and ready methods of the old examination. Secondly, they have not only produced new methods worthy of the old examiner's consideration, but, what is equally important, they have laid themselves open to criticism by publishing them, and have thus challenged the tradition of examinations as esoteric ceremonies controlled by a priestly caste.

These two factors have not remained independent, for there is now considerable collaboration between the psychologist and the L.E.A. A recent publication—*Northumberland Standardised Tests*, by Dr. Burt *—is a product of such collaboration. It is proposed to consider the chief principles underlying these tests mainly from the point of view of their possible application to examinations generally. Dr. Ballard has discussed these principles, in a general way, in his book *The New Examiner*.† The principles to be considered are:

(1) *Involved Questions are avoided.*—There are 156 short questions (to be done in 49 minutes). Dr. Burt draws attention to this as a special feature of the test. Dr. Ballard discusses it fully.

Short questions are not unknown in the class-room, and also sometimes appear in conventional arithmetic examinations as exercises to be worked mentally. But the possibilities of the short question had certainly not been fully exploited when the new examiner appeared and boldly decided that tests should consist *only* of a long series of short questions. In doing this he has raised an issue which should be faced by all interested in examining. Is it advisable to adopt the " short question " method, and, if so, to what extent ?

Dr. Ballard argues that the use in an examination of a comprehensive series of short questions tends to defeat the crammer, and he regards coaching as impossible for an intelligence test or an attainments test designed, as he thinks they should be, to test thinking power. It is quite true, as he says, that the " short question " method tends to prevent concentrated study on topics popular with examiners. But, on the other hand, the certainty that the

* University of London Press. † Hodder & Stoughton.

examination will cover thoroughly the whole syllabus may tend to encourage cramming in the form of concentration on the minimum portion of the syllabus necessary to ensure pass standard. And as to the impossibility of coaching for "thinking" questions, experience shows that constant training in tackling such questions increases the capacity for solving them much in the same way as constant contact with town life produces a more alert and adaptable mental attitude than does association with the slow moving life of the country. The point, however, has recently been the subject of experimental enquiry by A. E. Chapman,* who investigated the effect of special coaching on performance in intelligence tests of various kinds, including arithmetical problems and number series. He concluded tentatively that "in the case of children at school the results of intelligence tests may be affected by the nature of both general training and special coaching." He found the effect of special coaching most marked in the case of the number series. This is not surprising, as this test is the most unconventional of those he used.

It appears then that the "small question" method does not eradicate, or even reduce considerably, the cramming evil, and so its adoption cannot be urged on that account.

But further, there is a definite objection to the "small question." The solving of an involved arithmetical example necessitates the exercise of mental capacity in two aspects which are inadequately tested by small questions, viz. organisation and expression. The process of solution necessitates (*a*) holding all the associated ideas of the problem together in the mind and comprehending their relationship, *i.e.* conceiving the solution, (*b*) marshalling these ideas as a whole in their correct form, *i.e.* expressing their relationship, in a written solution.

Ballard argues that, as regards organisation, the thought processes used in solving the short question are essentially the same, though less consciously exercised. He does not press this point, however, and he suggests new types of exercises which should test the power to organise larger units. We need not therefore criticise his argument except to point out that the number of ideas which a mind can hold at one time is a measure of its potentiality; so also are the nature and degree of complexity of the relationships then apprehended.

With regard to "expression" Ballard defends his position by asking whether the old examiner can test it. He can certainly set tests for it, and, in mathematics, these are not hard to mark in a scientific way. So the reply is in the affirmative. But this is not the same as saying that he *does* test it adequately. This is certainly not true of one important examining authority, which considers that any solution of a problem leading to the correct answer should receive full marks—and this in a competitive examination. Nor is it true of any examiner who fails to assess method as well as result.

We reject then the idea of the all exclusive use of the short question. Is there any use for it at all, and, if so, how much? The Liverpool Association of Schoolmasters have recently passed a resolution that the papers in arithmetic (in the Scholarships and Free Place Examinations) should contain a greater number of questions than at present, and that these should be of a less involved character. So the point has already become a practical one in Liverpool. It seems worth the consideration of examiners generally.

(2) *Standardisation of Marking.*—This follows as a necessary corollary to the principle of setting innumerable short questions. An answer is either correct or incorrect and is marked accordingly, no other alternative being admitted. Thus objectivity in marking is attained.

We can appreciate fully the aim of the new examiner in attempting to eliminate the personal equation. One hears strange tales of the eccentricities

* *Forum* Nov. 1924.

of examiners in public examinations, and the well-known illustration by Starch [p. 434, *Educational Psychology*, D. Starch. 1920. Macmillan. The marks assigned by 114 mathematics teachers to a geometry script varied from 28 to 90] shows what great divergencies in marking can exist without definite standardisation. " Give good shots good marks "—we quote from experience —is hardly explicit enough to ensure that an assistant examiner should mark to standard. As to what degree of standardisation is desirable it is difficult to say. Where borderline cases are re-marked by one examiner the need is not so great as in other cases. It is possible to produce elaborate mark schemes by assessing each " point " of a solution, a " point " being an idea (*e.g.* a decision to multiply), a method (*e.g.* inversion of fraction for division) or the result of a calculation.

This method has the advantage of showing clearly the principles on which the chief examiner is marking, and so makes it easy for his assistants to interpret his policy in marking. It is slow at first, but becomes satisfactorily rapid as typical solutions recur. It has defects—a solution is sometimes more easily assessed by considering its faults rather than its positive achievements. But, on the whole, it is more reliable than standardising a few solutions and leaving the rest to the individual examiners concerned. It can be made as elaborate or as simple as one pleases.

There are always a number of general principles on which it is important for examiners to agree, apart from the details of marking. For instance, some examiners regard an error in simple calculation (*e.g.* $15-8$) as trivial, others as serious; others discriminate between the circumstances under which it may occur. Errors in misreading, or in misquoting tables, should be standardised. Fundamental errors which are to be regarded as vitiating a whole solution should be agreed upon.

The standardisation of marking brings its own evils, and the new examiner often secures uniformity of marking at the expense of efficiency. In Dr. Burt's *Northumberland Tests*, for instance, the intrinsic value of the work done, as indicated by the difficulty of the question or the amount of work involved in the solution, bears no relation to the mark assessment. Thus each correct figure in the subtraction $23-12$ carries one mark, while only one mark is awarded for a correct solution of a problem like this :

" Two boys start walking to meet one another from two towns $10\frac{1}{2}$ miles apart. If one walks at 3 miles an hour and the other at 4 miles an hour, how long will it be before they meet ? "

No provision is made for the possibility of a correct method being used when the answer is incorrect ; nor is there any assessment for the power of expressing the method of solution on paper, answers alone being required.

(3) *Standardisation of Tests.*—The new examiner tries his tests beforehand, and so has some measure of the results to be expected. He is also thus able to eradicate unsuitable questions or reword those unsuitably worded. That he does not then necessarily attain absolute perfection is shown by this question of Dr. Burt's :

" Mr. Aaron Abrahams has signed his name eight times this morning. In doing so, how many times has he written the first letter of the alphabet ? "

But he generally avoids the unreasonably difficult or abstruse example which the old examiner (including the writer, alas !) has sometimes produced, because he is not content to try his questions on a friend or two, or even on a committee of adults, but on numerous groups of pupils of normal ability. His whole procedure, however, is extremely elaborate and need not be detailed.

But the underlying idea, viz. that an examination should have, for normal pupils, a definite Achievement Standard, known to and determined by the examiner beforehand, is one which cannot be lightly dismissed.

The idea in one form has already found its way into schools, particularly in U.S.A., where the measurement of educational results generally is carried

on with much thoroughness. Thus many standardised tests in mathematics are already being utilised in secondary schools for various purposes, *e.g.* the Rugg-Clark tests in First Year Algebra. In England, the use in elementary schools of standardised tests, such as those of Dr. Burt, is contemplated in the recent *Board of Education Report on Psychological Tests*, and they are in fact already being used freely. Is this movement going to spread and culminate in a demand that the old examiner shall come into line and standardise his examinations ? Will it be required eventually, for instance, that in all school certificate examinations the mathematics paper shall be of the same predetermined order of difficulty. Some L.E.A.'s already publish norms of performance of the candidates in their examinations, but their mark schemes are not made public. Publication of both norms and mark scheme is necessary if examinations are to be fully subject to the criticism of public opinion and also to be of subsequent use to teachers in ascertaining whether their classes have attained required standards. Should mark schemes be published ?

The fetish of examinations already looms too large in most schools, so that, taking a broad educational point of view, much of the energy spent on attempting to refine existing examination methods would seem misdirected. Specific danger in the movement too is shown by the absurd lengths to which the use of standardised tests is carried in Secondary Schools in U.S.A.*

The general tendency, however, is evident, and, whatever its ultimate outcome, it may, at any rate, have value to the old examiner at the moment in suggesting whether, from the point of view of his own job, he can do more towards safeguarding his examination papers from defects and ensuring that they anticipate a proper standard of attainment.

(4) *Intelligence and Attainments.*—The new examiner tests these two qualities separately, and, in case of the latter, the test may again be subdivided into various topics. Thus Dr. Burt has sections dealing with : Addition, Subtraction, Multiplication, Division, Mental Arithmetic, Rules, Reasoning.

These principles are not new, especially in junior examinations, but the separation has generally not been carried out so thoroughly by the old examiner. In fact, it is not uncommon to find in conventional examinations problems involving both complicated mechanical work and hard thinking. Many an unfortunate candidate must have carried through a lengthy process of calculation with fruitless results owing to a mistake in reasoning in the early stages. The new examiner avoids this type of question, and, bearing in mind the difficulty in assessing solutions when incorrect, we think his method makes for justice to the candidates.

On the whole there are several features worth attention in modern examination methods, and the old examination will no doubt reflect these in due course. In fact it has already begun to do so, junior examinations particularly.

R. S. Williamson.

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403. Draw a circle, diameter 85. $ABCD$ is an inscribed quadrilateral, BD cutting AC in I. $AB=75$, $BC=68$, $CD=60$.

A quadrilateral $ABCD$ is inscribed in a circle, diameter 85, and BD cuts AC at right angles in I. $AB=75$, $BC=68$, $CD=40$, $BI=60$, $DI=24$, $AI=45$ and $CI=32$. As nearly as possible A represents Legnago, B Mantua, C Peschiera, and D Verona—the fortresses of the famous Italian "Quadrilateral."—Sphinx-Oedipus, Oct. 1913.

404. Mersenne joined in 1612 the religious order of the Minimes. Voltaire nicknamed him "le minime et très minime Pere Mersenne."

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* *U.S.A. Mathematical Association Report on the Reorganisation of Mathematics in Secondary Education.*

"INTELLIGENCE" AND SCHOOL EXAMINATIONS.

BY MISS O. M. STANTON, M.A.

"GENERAL Intelligence" is a will-o'-the-wisp, and Psychologists have been chasing it with Tests for a considerable time. Whether it has yet been brought to heel and can be exactly measured is still open to question, but the modern Intelligence Tests certainly discover some kind of innate ability. Is this ability purely "general," or do some types of ability show up better than others in the Tests ?

The investigations described in this note are an attempt to discover whether high "intelligence" is correlated with success in Arithmetic, History, or English, or an average of all three. I say "success" advisedly, for the only means of testing mathematical or literary ability have been the marks obtained in ordinary school examinations, and these obviously test attainments rather than ability. Examination results (attainments) are immediately affected by illness, absence from school, laziness, and innumerable other circumstances, so that a child of good ability may by some accident be low on the list. Nevertheless, school examinations are a useful working method for the discovery of ability, and are likely to remain the sole method of testing in many schools for a long time to come. It seems worth while, therefore, to find out whether ability measured by this means has any close relation to the "intelligence" of the "Intelligence Tests."

The numbers involved in these investigations are very small, so small as to be quite useless, statistically, as a basis for any general conclusion. I put them forward, however, in the hope that they may be supported, or perhaps refuted, by the results obtained in other schools.

The *data* used in these investigations are as follows:

Subjects. The tests were applied to a form of about 30 girls, average age 12 years, in a large Secondary School. The girls entering the schools are drawn chiefly from Elementary Schools, and the majority of girls in the form tested (an Upper Third form) had been in the school for about six months before the Intelligence Test was applied.

Tests. The Intelligence Test used was the Northumberland Standardised Test, revised by Dr. Cyril Burt, 1925.*

The Arithmetic, English and History lists were the orders of merit in the ordinary end-of-term school examinations (December 1925 and July 1926; six lists in all).

Method. On the results of the Northumberland Tests, the children were arranged in order of merit in two ways:

(*a*) According to the actual marks obtained in the test by each child; called in these notes "list (*a*)."

(*b*) According to the "Intelligence Quotient" of each child; called in these notes "list (*b*)."

The Intelligence Quotient (I.Q.) was obtained by dividing the marks obtained by each child, by the "norm" for her age, and multiplying the result by 100. These norms are given in the *Book of Instructions* (p. 12) issued with the Tests.

Thus list (*a*) ranks the child according to her ability regardless of age; list (*b*) puts at the top the bright young child, who may have been half-way down list (*a*); and the "grandmother" of 14, near the top of list (*a*) settles down at the bottom of list (*b*) with an I.Q. of about 90. In this particular form, however, the children were of average ability, with very few outstanding cases at the top or bottom. The two lists (*a*) and (*b*) were therefore very similar: in fact, the correlation coefficient between them was as high as ·89.

* Published by the University of London Press. See also Burt's *Mental Scholastic Tests*, pages 221 ff.

The other lists used will be called for convenience A_1 and A_2 (order of merit in Arithmetic Examinations in December 1925 and July 1926 respectively); E_1 and E_2 (order in English Examinations); H_1 and H_2 (order in History Examinations); and finally M, the order of merit on an average of the six lists.

The relation between two arrangements is best disclosed by a correlation coefficient, and for work of this kind Pearson's coefficient is usually the most convenient, viz.

$$\sigma = 1 - \frac{6\Sigma d^2}{n(n^2-1)},$$

where n is the number of names on each list, d the rank differences of the subjects in the two lists.

For brevity, only a short imaginary example will be given here to illustrate the method of calculation:

Name.	Rank in list (*a*).	Rank in list *A*.	Rank Difference.	Square of Difference.
AB	1	3	2	4
CD	2	2	0	0
EF	3	1	2	4
GH	4	5	1	1
KL	5	6	1	1
MN	6	4	2	4

$$\Sigma d^2 = 14$$

$$\begin{aligned}\sigma &= 1 - \frac{6\,.\,14}{6\,.\,5\,.\,7}\\ &= 1 - \cdot 4\\ &= \cdot 6.\end{aligned}$$

It is obvious that the maximum correlation takes place when the two lists are identical, *i.e.* when $\Sigma d^2 = 0$. The maximum value is therefore 1.

Similarly, when one list is the exact reverse of the other,

$$\sigma = -1,$$

for in this case
$$\begin{aligned}\Sigma d^2 &= (n-1)^2 + (n-3)^2 + \text{etc.}\\ &= \frac{n(n^2-1)}{3};\end{aligned}$$

and when the lists have no relation whatever,

$$\sigma = 0,$$

for in this case
$$\Sigma d^2 = \frac{n(n^2-1)}{6}.$$

A coefficient greater than ·3 may be considered "significant," and greater than ·5 "large."

The lists themselves are too cumbersome to be reproduced here; all that is of interest for our purpose is the correlation between them. The following table gives the various coefficients:

	A_1	A_2	E_1	E_2	H_1	H_2	M
a	·20	·37	·21	·19	·13	·09	·31
b	·12	·39	·21	·07	·25	·05	·22

Interpretation of these results:

(1) The most noticeable feature of these coefficients is their smallness; all are positive, but only three are significant.

(2) Secondly, the only significant coefficients, apart from the list (M), are both Arithmetic.

How are these two features to be accounted for?

(1) The *low correlation* seems to indicate that the so-called Intelligence Test does not discover in the children ability to do well in School Examinations. Are we to conclude at once either that intelligence is not required for examination purposes or that the Northumberland Tests are not true Tests of Intelligence? Fortunately, we need not accept either of these conclusions.

Owing to the fact that the negative term in the coefficient contains the *squares* of differences of rank, the correlation is quickly brought down by two or three large differences. If the number of children dealt with had been large a few exceptional cases would have been swallowed up, but with so small a number as 30 they are sufficient to affect the correlation considerably. A close examination of individual girls does, in fact, reveal adequate reasons in most cases for the large differences of rank in different lists, and if three or four exceptional cases are omitted very significant coefficients are obtained.

We will take first the case of *AB*, an apparently intelligent hard-working girl, who ranks high in most school examinations, and very high in some. In list (*a*), however, she ranks 22nd; and in list (*b*) 24½. Her I.Q. is only 96. These differences are possibly accounted for by the fact that at the time of the Intelligence Test she was living through trying domestic circumstances, was tired and worried, and not quite able to do her best. Moreover, she has apparently been obtaining help from other girls during some school examinations—a feat which is practically impossible in an Intelligence Test. Finally, hard work is a supplement to, though not a substitute for, intelligence, and she is a hard worker.

CD ranks high in English, History and one Arithmetic list; she is 27th in list (*a*), 26½ in list (*b*), and her I.Q. is 92—very low for a Secondary School. She, again, is a hard worker, but is rather "slow in the up-take," and is of a shy and nervous temperament; in the Intelligence Test she was probably over-anxious to do well, and, realising that "speed" was important, was flustered, and did not do herself justice.

EF is 25th in Arithmetic, 9th in list (*a*), 4th in list (*b*), and has an I.Q. of 118. She is thoroughly lazy, and seldom does any homework. That is quite enough to explain the differences in rank.

GH is much older than the rest of the form; she has fairly good intelligence (I.Q. 102), but does badly in Arithmetic and History. In her case, illness and very long absences from school in early childhood account for the lack of grounding in Arithmetic and other subjects which she is still trying to make up.

In fact, in almost every case of large differences of rank in the Test and Examination lists, adequate physical or psychological reasons are apparent. Where intelligence is revealed by the Tests but the subject does badly at school, there are almost always commonsense reasons for the discrepancy.

Complete correlation cannot, of course, be expected, for the Test and the Examination aim at measuring different things—the one ability, the other attainments—and ability is only one of many factors influencing a child's attainments.

(2) The correlation between one Arithmetic list and the Test lists is much greater than any of the others. Are we to conclude that the "Intelligence" tested is not quite general, but has a bias towards the special ability needed for Mathematics? A closer examination of individual children is again of interest here.

The child *AB* mentioned before was very high in Arithmetic, but low in the Test. Now several Test questions need the exercise of reasoning power, others (*e.g.* " Opposites ") need chiefly a good vocabulary. In Question VI., " Selecting Reasons," *AB* obtained higher marks than any other child in the form, in all other questions she obtained low marks.

Again, *KL* was placed high in Arithmetic and in the Test, low in English and History. Her marks in the Test were gained mainly on the questions involving reasoning and " following an argument " ; her vocabulary was weak.

MN had good positions in History and English, but came low down in Arithmetic and in the Test. She obtained good marks, however, in Questions II., III., and V. of the Test (Opposites, Similarities and Completing Sentences), but very poor marks in " Simple Reasoning " and " Detecting Absurdities."

These three cases are typical of many, and in my opinion point to a weakness in the Northumberland Tests. At the age of 12, memory is an important part of one's mental equipment, but memory is scarcely tested at all in these tests. Moreover, at this age, a good deal of the work in History and Literature is a matter of pure memory, to a great extent simply rote memory.

It may be almost inevitable that rote memory should escape the examiner's net in Group Tests, but it figures prominently in individual tests, such as the Binet-Simon—perhaps rather too prominently. Considering the importance of this faculty in children, it seems desirable that some effort should be made to test it in the ordinary Group Tests.

Further, we should test the ability to read silently, to grasp the meaning of what is read sufficiently well to reproduce it in one's own words, or to show in any other way that what is read is understood and remembered. These are faculties important in all subjects, and especially so in the so-called " English subjects," but these are again, in my opinion, insufficiently tested in the Group Tests. The Tests VIII. and IX., " Following an Argument " and " Detecting Absurdities," involve, it is true, reading and understanding, but, I think, are inadequate. Also, so many mental processes are involved in each of them that it is difficult to tell exactly what is being tested.

Summary of Conclusions. Owing to the very small scale of these investigations, I draw my conclusions very diffidently. They are, however, these :

(1) That the Northumberland Intelligence Tests do discover the subjects' innate ability, more or less independently of previous education or opportunity.

(2) That Mathematical (Arithmetical) ability is more highly correlated with " Intelligence " than ability in " English subjects " ; at least in the early years at the Secondary School ; but

(3) That this may be due to the fact that memory is practically ignored in these tests, and plays a very important part in the early teaching of " English subjects."

In spite of the correlation coefficients, I do not feel justified in concluding that mathematicians have on the whole higher Intelligence Quotients than the common run of men ; before any such conclusions can be ventured, these statistics must be multiplied at least a hundredfold.

It cannot be too clearly emphasised that the bases on which these conclusions rest are so small as to render them valueless, unless supported by similar results obtained by many other teachers. Our psychologists have investigated on an extensive scale, and have established invaluable results which it seems arrogant to criticise. Nevertheless, I feel that it is one's business as an ordinary teacher, not merely to bow before them as they pass, but to follow after them. If the " ordinary teacher " will take the trouble to make a few calculations for himself and compare his results with others, it is probable that the conclusions of the great investigators will be n[illegible]ly established, but extended.

O. M. Stanton.

MATHEMATICAL NOTES.

856. [A. 1. b.] *Proof of the theorem that the A.M. of n quantities exceeds their G.M.*

In C. Smith's *Algebra* (1910), p. 439, there is a direct proof of the theorem that if m and r be positive and $m>r$; then, unless $a_1=a_2=a_3=$etc.,

$$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n}>\frac{a_1^r+a_2^r+\ldots+a_n^r}{n}\times\frac{a_1^{m-r}+a_2^{m-r}+\ldots+a_n^{m-r}}{n}.$$

It runs thus:

"We have to prove that

$$n(a_1^m+a_2^m+\ldots)>(a_1^r+a_2^r+\ldots)(a_1^{m-r}+a_2^{m-r}+\ldots);$$

or that $$(n-1)(a_1^m+a_2^m+\ldots)>\Sigma(a_1^r a_2^{m-r}+a_1^{m-r}a_2^r);$$

or that $$\Sigma(a_1^m+a_2^m-a_1^r a_2^{m-r}-a_1^{m-r}a_2^r)>0,$$

every letter being taken with each of the $n-1$ other letters.

Now $a_1^m+a_2^m-a_1^r a_2^{m-r}-a_1^{m-r}a_2^r=(a_1^r-a_2^r)(a_1^{m-r}-a_2^{m-r})$,

which is positive since $a_1^r-a_2^r$ and $a_1^{m-r}-a_2^{m-r}$ are *both* positive or *both* negative according as a_1 is greater or less than a_2. Hence

$$\Sigma(a_1^m+a_2^m-a_1^r a_2^{m-r}-a_1^{m-r}a_2^r)>0,$$

which proves the proposition."

From this it is clear that a repeated application gives

$$\frac{a_1^m+a_2^m+\ldots}{n}\geqq\frac{a_1+a_2+\ldots}{n}\cdot\frac{a_1^{m-1}+a_2^{m-1}+\ldots}{n}$$

$$\geqq\frac{a_1+a_2+\ldots}{n}\cdot\frac{a_1+a_2+\ldots}{n}\cdot\frac{a_1^{m-2}+a_2^{m-2}+\ldots}{n}\geqq\ldots$$

$$\geqq\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m.$$

And this is true, however great m may be; for:

I. If a_1, a_2 $(a_1>a_2)$ are both *less* than unity, then

$$\operatorname*{Lt}_{m\to\infty}(a_1^r-a_2^r)(a_1^{m-r}-a_2^{m-r})=0.$$

Hence, if *all* the a's are less than unity, we have

$$\operatorname*{Lt}_{m\to\infty}\left[\frac{a_1^m+a_2^m+\ldots+a_n^m}{n}-\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m\right]=0.$$

II. If a_1, a_2 $(a_1>a_2)$ are both *greater* than unity, then

$$\operatorname*{Lt}_{m\to\infty}\frac{1}{a_1^{m-r}-a_2^{m-r}}=\operatorname*{Lt}_{m\to\infty}\frac{1}{a_1^{m-r}\left[1-\left(\frac{a_2}{a_1}\right)^{m-r}\right]}=\operatorname*{Lt}_{m\to\infty}\frac{1}{a_1^{m-r}}=0.$$

Hence, by increasing m we can make the expression

$$(a_1^r-a_2^r)(a_1^{m-r}-a_2^{m-r})$$

as large as we please. Thus, if some of the a's are greater than unity, then, no matter how great m may be, the expression

$$\frac{a_1^m+a_2^m+\ldots+a_n^m}{n}$$

will never be less than the expression

$$\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m.$$

Now let $a_r = \log_e a^r$. Then

$$\frac{1}{n}\sum_{r=1}^{n}(\log_e a_r)^m \geqq \left(\frac{1}{n}\sum_{r=1}^{n}\log_e a_r\right)^m.$$

And generally,

$$1+\frac{1}{n}\sum_{m=1}^{\infty}\sum_{r=1}^{n}\frac{(\log_e a_r)^m}{m!} \geqq 1+\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{1}{n}\sum_{r=1}^{n}\log_e a_r\right)^m,$$

i.e.

$$\frac{\sum_{r=1}^{n} e^{\log_e a_r}}{n} \geqq e^{\frac{1}{n}\sum_{r=1}^{n}\log_e a_r},$$

or

$$\frac{a_1+a_2+\ldots+a_n}{n} \geqq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}.$$

Hence *the A.M. of n quantities is never less than their G.M.*

ARTHUR J. CARR.

857. [L[1]. 9. d.] *Note on Expression for Arc of Ellipse.*

In the October issue (Note 855, *Gazette*, xiii. p. 206), Mr. Milward puts forward a geometrical method of determining the ratio of the perimeter of an ellipse to the major axis, the validity of which seems to be very doubtful.

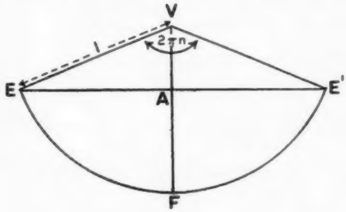


FIG. 1.

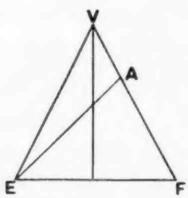


FIG. 2.

In effect, he says: Let Fig. 1 be a sector of a circle and Fig. 2 represent the cone formed by folding the sector until VE' and VE coincide, so that point A takes up position A in Fig. 2; then consider a section of this cone through the points E and A (in Fig. 2) whose vertical trace is EA; the perimeter of the elliptic section so formed will then be equal in length to the str. line EAE' in the development of the cone in Fig. 1.

It is this last assumption which constitutes the fallacy in his method.

Firstly, it may easily be seen that the tangent to the perimeter of the elliptic section at E in Fig. 2 is $\perp EA$ and VE, and hence the development of the perimeter in Fig. 1 must be a curve which is $\perp VE$ at E and $\perp VE'$ at E', a fact which rules out the possibility of the development being the str. line EE'.

Secondly, if the development of the section is traced out corresponding to Mr. Milward's figure in the case when $n=\frac{1}{4}$, it will be found to be a curve similar to the one shown dotted in Fig. 4.

Thirdly, it would appear that Mr. Milward has considered a particular section of his cone, but the question as to whether its development is a str. line, or

indeed as to whether the development of *any* section of a cone can ever be a str. line, could be settled by finding the polar eqn. of the development.

Thus:

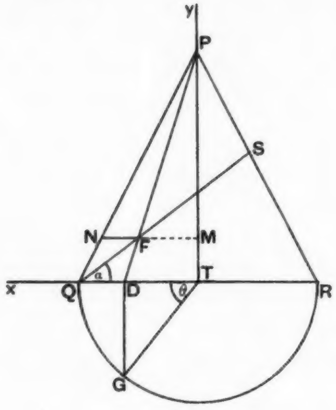


FIG. 3.

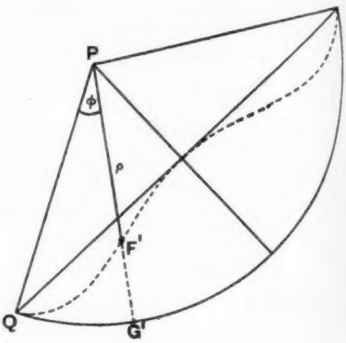


FIG. 4.

Let PD be the elevation of any generator and TG its plan. Then if QS be the elevation of the trace of the section, PN will be the radius vector in the polar development corresponding to this generator, where $FN \parallel QT$. Let $QT=a$, $TP=b$, $Q\hat{T}G=\theta$ and polar coords. of F' be (ρ, ϕ). Then

$$\text{arc } QG = \text{arc } QG' \quad \text{or} \quad a \cdot \theta = \sqrt{a^2+b^2} \cdot \phi.$$

Referred to Ox, Oy, eqn. of QS is $y=-x\tan\alpha + a\tan\alpha$,

$$\text{eqn. of } PD \text{ is } \frac{x}{a\cos\theta}+\frac{y}{b}=1.$$

Solving these eqns. we get $TM=\dfrac{ab\tan\alpha(\cos\theta-1)}{a\cos\theta\tan\alpha-b}$;

$$\therefore \; MP=b-TM=\frac{-b^2+ab\tan\alpha}{a\cos\theta\tan\alpha-b}.$$

Also eqn. of QP is $x/a+y/b=1$. Solving this eqn. with

$$y=TM=\frac{ab\tan\alpha(\cos\theta-1)}{a\cos\theta\tan\alpha-b}$$

to get MN, we obtain

$$\frac{x}{a}+\frac{a\tan\alpha(\cos\theta-1)}{a\cos\theta\tan\alpha-b}=1\,; \quad \therefore \; x=\frac{a(a\tan\alpha-b)}{a\cos\theta\tan\alpha-b}\;(=MN)\,;$$

$$\therefore \; PN^2=NM^2+MP^2$$

$$=\frac{a^2(a\tan\alpha-b)^2}{(a\cos\theta\tan\alpha-b)^2}+\frac{b^2(-b+a\tan\alpha)^2}{(a\cos\theta\tan\alpha-b)^2}\,;$$

$$\therefore \; PN=\frac{(a\tan\alpha-b)\sqrt{a^2+b^2}}{a\cos\theta\tan\alpha-b}.$$

$\therefore$ polar eqn. of development is

$$\rho=\frac{(a\tan\alpha-b)\,.\,\sqrt{a^2+b^2}}{a\tan\alpha\,.\,\cos\left(\dfrac{\sqrt{a^2+b^2}}{a}\,.\,\phi\right)-b}\ \ldots\ldots\ldots(1),\text{ using } a\theta=\sqrt{a^2+b^2}\,.\,\phi.$$

It is fairly clear that this eqn. could not represent a str. line, except in the limiting case when the cutting plane is a tangent plane to the cone, for, owing to the coefficient of ϕ, this eqn. could not be put in the form $\rho\cos\phi=$ a const.

Furthermore, if we write this eqn. in the form

$$\rho=\frac{A}{B\cos K\phi+C},$$

where A, B, K, C do not involve ϕ, and find $\rho\,.\,\dfrac{d\phi}{d\rho}$, we get

$$\tan\chi=\frac{(B\cos K\phi+C)}{BK\sin K\phi},$$

where χ is the angle between tangent and radius vector.

Hence, if $\phi=0$, then $\chi=\pi/2$, and therefore the development meets PQ at rt. angles, from which it follows that there is no section of any cone whatever the trace of which would develop into a str. line.

There seems to be little point in proceeding to consider that particular section chosen by Mr. Milward. As a matter of interest, however, the polar eqn. of his section can easily be deduced in terms of the symbols of his note as follows:

$$\text{Put } a=nl,\ \sqrt{a^2+b^2}=l,\ b=l\sqrt{1-n^2}.$$

Use the Sine Rule in $\triangle EAF$ (Fig. 2) to get

$$\sin\alpha=\frac{l(1-\cos n\pi)\sqrt{1-n^2}}{\sqrt{l^2-2l^2\cos n\pi+l^2\cos^2 n\pi+4n^2l^2\cos n\pi}},$$

which gives
$$\tan\alpha=\frac{\sqrt{1-n^2}}{n}\cdot\frac{1-\cos n\pi}{1+\cos n\pi}.$$

If we substitute these values in (1), we get, after some reduction,

$$\rho=\frac{2l\cos n\pi}{1+\cos n\pi-\cos\dfrac{\phi}{n}(1-\cos n\pi)},$$

whilst the polar eqn. of the str. line EE' (Fig. 1) is $\rho\cos(n\pi-\phi)=l\cos n\pi$.

H.M. Dockyard School, Devonport. V. Naylor.

858. [E. 1. e.] *Note on Stirling's Theorem.*

The following, though non-rigorous and in the second part empirical, may be of interest to some readers.

(1) If x is an integer, and n a finite integer,

$$\lim_{x\to\infty}\frac{\lfloor x+n}{x^n\lfloor x}=1\,;$$

and this still holds true for all values of x and n such that neither x nor $x+n$ is a negative integer, if we generalise the factorial, replacing $\lfloor m$ by $\Pi(m)$ or $\Gamma(m+1)$.

Hence putting $n=-\frac{1}{2}$, and remembering that $\lfloor -\frac{1}{2} = \sqrt{\pi}$,

$$\frac{\lfloor x-\frac{1}{2}}{\lfloor x} = \frac{(x-\frac{1}{2})(x-\frac{3}{2})\dots\frac{1}{2}}{x\,.\,(x-1)\dots 1}\,.\sqrt{\pi}$$

$$= \frac{(2x-1)(2x-3)\dots 3\,.\,1}{2x(2x-2)\dots 4\,.\,2}\,.\sqrt{\pi}$$

$$= \frac{\lfloor 2x}{(\lfloor x)^2\,.\,2^{2x}}\,.\sqrt{\pi}.$$

Hence
$$\lim_{x\to\infty} \frac{\lfloor 2x}{(\lfloor x)^2}\,.\,\frac{\sqrt{\pi x}}{2^{2x}} = 1\,;$$

or, approximately, if x is large,

$$\frac{\lfloor 2x}{(\lfloor x)^2} = \frac{2^{2x}}{\sqrt{\pi x}}\,. \quad\dots\dots\dots\dots(a)$$

Denoting $\frac{\lfloor 2x}{(\lfloor x)^2}$, $\frac{2^{2x}}{\sqrt{\pi x}}$ by A, B respectively, we have by the use of logarithmic tables,

x	$\log A$	$\log B$	B/A	$1+\frac{1}{8x}$
10	5·26660	5·27212	1·0128	1·0125
15	8·19065	8·19424	1·0084	1·0083
20	11·13939	11·14211	1·00628	1·00625
25	14·10178	14·10396	1·00502	1·00500
30	17·07285	17·07467	1·00419	1·00417
35	20·04994	20·05149	1·00358	1·00357

A comparison of the two last columns leads to

$$\frac{\lfloor 2x}{(\lfloor x)^2} = \frac{2^{2x}}{\sqrt{x\pi}}\Big/\left(1+\frac{1}{8x}\right). \quad\dots\dots\dots\dots(b)$$

(2) Since
$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\frac{(1+x)^x}{x^x},$$

$$\therefore\ \lim_{x\to\infty}(x+1) = \lim_{x\to\infty}\frac{(x+1)^{x+1}e^{-\overline{x+1}}}{x^x e^{-x}}\,;$$

$\therefore$ for n finite,
$$\lim_{x\to\infty}\frac{\lfloor x+n}{\lfloor x} = \lim_{x\to\infty}\frac{(x+n)^{x+n}e^{-\overline{x+n}}}{x^x\,.\,e^{-x}}.$$

We therefore assume that approximately

$$\lfloor N = C\,.\,N^{N+\alpha}\,.\,e^{-N+\beta}\left(1+\frac{\gamma}{N}\right);$$

and we have
$$\frac{\lfloor 2N}{(\lfloor N)^2} = \frac{C(2N)^{2N+\alpha}e^{-2N+\beta}\left(1+\frac{\gamma}{2N}\right)}{C^2N^{2N+2\alpha}e^{-2N+2\beta}\left(1+\frac{2\gamma}{N}+\frac{\gamma^2}{N^2}\right)}$$

$$= \frac{2^{2N+\alpha}}{C}\,.\,\frac{1}{N^\alpha}\,.\,e^{-\beta}\Big/\left(1+\frac{3\gamma}{2N}\right) \text{ nearly.}$$

Comparing this with, from (*b*) above,

$$\frac{\lfloor 2N}{(\lfloor N)^2}=\frac{2^{2N}}{\sqrt{\pi}}\cdot\frac{1}{N^{\frac{1}{2}}}\Big/\left(1+\frac{1}{8N}\right),$$

we have $\alpha=\frac{1}{2}$, $\beta=0$, $\gamma=\frac{1}{12}$, $C=\sqrt{2\pi}$.
Hence, if N is large,

$$\lfloor N=\sqrt{2\pi}\,.\,N^{N+\frac{1}{2}}\,.\,e^{-N}\left(1+\frac{1}{12N}\right)\text{ nearly.}$$

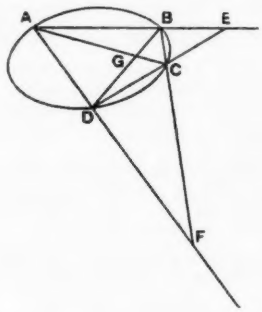
J. M. CHILD.

859. [L[1]. 17. a.] *To find the equation of the other two pairs of lines through the four points determined by the intersection of the conic $S=0$ with the lines $L=0$ and $L'=0$.*

In the figure, the line AB is $L=0$ and the line CD is $L'=0$. The polar of E, their point of intersection, passes through F and G, which are the centres of the conics given by

$$\lambda S-2LL'=0 \quad\text{.....................................(1)}$$

when this equation represents straight lines.



The polar of E with regard to $S=0$ is

$$\begin{vmatrix} X & Y & Z \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0. \quad\text{.................................(2)}$$

The centre of (1) is given by

$$\lambda X=lL'+l'L, \quad\text{.......................................(3)}$$

$$\lambda Y=mL'+m'L. \quad\text{..................................(4)}$$

The elimination of x and y between (2), (3) and (4) will accordingly give the two values of λ required.

From (3) and (4),
$$\frac{1}{\lambda}=\frac{X}{lL'+l'L}=\frac{Y}{mL'+m'L};$$

each ratio $=\dfrac{Z}{nL'+n'L}$ by (2).

$$\therefore\ \frac{1}{\lambda}=\frac{pX+qY+rZ}{p(lL'+l'L)+q(mL'+m'L)+r(nL'+n'L)}.$$

Putting $p=Al+Hm+Gn$, $q=Hl+Fm+Bn$ and $r=Gl+Fm+Cn$,

$$\frac{1}{\lambda}=\frac{\Delta L}{\Sigma L'+\Pi L}. \quad \text{.................................}(5)$$

Putting $p=Al'+Hm'+Gn'$, etc.,

$$\frac{1}{\lambda}=\frac{\Delta L'}{\Pi L'+\Sigma' L}. \quad \text{.................................}(6)$$

From (5) and (6), $(\lambda\Delta-\Pi)L=\Sigma L'$

and $(\lambda\Delta-\Pi)L'=\Sigma' L.$

Hence $(\lambda\Delta-\Pi)^2=\Sigma\Sigma'$,

and from (1) the equation of the two pairs of lines is

$$(2LL'\Delta-\Pi S)^2=\Sigma\Sigma' S^2.$$

N. M. Gibbins.

860. [V. 1. a. λ.] *The Approach to the Logarithmic and Exponential Functions.*

In the recent discussion on this subject, the interesting theorem stated by Mr. W. J. Dobbs in the January 1910 number of the *Gazette*, page 179, seems to have been overlooked, but the following modification of that theorem seems to offer the simplest approach for pupils of medium ability:

(1) Prove that if $y=a^x$, then $\frac{dy}{dx}=ky$ (as in Mr. Child's article), and thus establish the constancy of the sub-tangent for the curve.

(2) Show that the tangents from the origin to the family of curves $y=a^x$ touch the curves in points whose ordinates are constant, thus introducing the idea of an absolute constant in relation to this family of curves. Call this constant "e."

(3) Now considering the special curve $y=e^x$, the abscissa corresponding to the ordinate e is 1. Thus the length of the sub-tangent for this particular curve is unity, and therefore $\frac{dy}{dx}=y$.

(4) Use this equation to obtain the series for e^x, employing such amount of rigour as is suitable to the class. Hence obtain

$$e=1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\text{etc.}$$

Mr. Katz' very clever method seems to me too difficult for the ordinary student, and his method of obtaining the value of e as the limit is not only non-rigorous, but is also too artificial, as is shown by the fact that it is impossible for even a good pupil to reproduce it without having it well drilled into him. The geometrical proof of Gregory of St. Vincent's theorem is very valuable for VI. Form pupils, and provides a simple and natural introduction to Hardy's definition of $\log x$ as $\int_1^x \frac{dt}{t}$, although it cannot take the place of that definition. My particular difficulty in starting with Hardy's definition has been the difficulty of connecting his treatment with the previous work of the pupil, but the end of the article by Mr. Katz supplies some solution of this difficulty.

D. K. Picken.

861. [X.] *Some approximate Circle Squarings.*

Being confronted recently with a circle squarer, I had occasion to work out some approximations which it may be of interest to record. The series refer to a square, equilateral triangle, regular hexagon, and a rhombus whose angle is 60°, each equal in area to a given circle. If d is the diameter of the circle, a the side of the polygon, we have $\pi d^2=4a^2$, $\sqrt{3}a^2$, $6\sqrt{3}a^2$, and $2\sqrt{3}a^2$ respectively. The values of a/d were expressed as continued fractions, and their

principal and intermediate convergents obtained. The final table gives the percentage errors of the areas corresponding to the principal convergents, the error being positive when the approximation is in defect.

The calculations were carried out by one of my students, Mr. L. Shotlander, B.A.

Square. $\frac{1}{2}\sqrt{\pi}=0{\cdot}88622\ 69255=0+\frac{1}{1}\,{}_{+}\frac{1}{7}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{3}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{2}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{57}\,{}_{+}\,.$

$$\text{Defect: } \frac{0}{1},\ \left(\frac{1}{2},\ \frac{2}{3},\ \ldots,\ \frac{6}{7}\right),\ \frac{7}{8},\ \left(\frac{15}{17},\ \frac{23}{26}\right),\ \frac{31}{35},\ \left(\frac{70}{79}\right),\ \frac{109}{123}.$$

$$\text{Excess: } \frac{1}{0},\ \frac{1}{1},\ \frac{8}{9},\ \frac{39}{44},\ \frac{148}{167}.$$

Equilateral triangle.

$$\sqrt{\frac{\pi}{\sqrt{3}}}=1{\cdot}34677\ 36865=1+\frac{1}{2}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{7}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{127}\,{}_{+}\,.$$

$$\text{Defect: } \frac{0}{1},\ \frac{1}{1},\ \frac{4}{3},\ \frac{35}{26},\ \frac{101}{75}.$$

$$\text{Excess: } \frac{1}{0},\ \left(\frac{2}{1}\right),\ \frac{3}{2},\ \left(\frac{7}{5},\ \frac{11}{8},\ \ldots,\ \frac{27}{20}\right),\ \frac{31}{23},\ \frac{66}{49},\ \frac{167}{124}.$$

Regular hexagon.

$$\sqrt{\frac{\pi}{6\sqrt{3}}}=0{\cdot}54981\ 80552=0+\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{4}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{13}\,{}_{+}\frac{1}{5}\,{}_{+}\frac{1}{3}\,{}_{+}\,.$$

$$\text{Defect: } \frac{0}{1},\ \frac{1}{2},\ \frac{6}{11},\ \left(\frac{17}{31},\ \frac{28}{51},\ \ldots,\ \frac{138}{251}\right),\ \frac{149}{271}.$$

$$\text{Excess: } \frac{0}{1},\ \frac{1}{1},\ \left(\frac{2}{3},\ \frac{3}{5},\ \frac{4}{7}\right),\ \frac{5}{9},\ \frac{11}{20}.$$

Rhombus (angle 60°). $\sqrt{\frac{\pi}{2\sqrt{3}}}=0{\cdot}95231\ 28065=0+\frac{1}{1}\,{}_{+}\frac{1}{19}\,{}_{+}\frac{1}{1}\,{}_{+}\frac{1}{32}\,{}_{+}\,.$

$$\text{Defect: } \frac{0}{1},\ \left(\frac{1}{2},\ \frac{2}{3},\ \ldots,\ \frac{18}{19}\right),\ \frac{19}{20},\ \left(\frac{39}{41},\ \frac{59}{62},\ \ldots,\ \frac{639}{671}\right),\ \frac{659}{692}.$$

$$\text{Excess: } \frac{1}{0},\ \frac{1}{1},\ \frac{20}{21}.$$

	a	d	Error %		a	d	Error %
Square	7	8	+2·5	Equilateral triangle	4	3	+2·0
	8	9	−0·60		31	23	−0·16
	31	35	−0·12		35	26	+0·092
	39	44	−0·031		66	49	−0·025
	109	123	+0·0097		101	75	+0·016
	148	167	−0·00014		167	124	−0·000074
Regular hexagon	5	9	−2·1	Rhombus	19	20	+0·48
	6	11	+1·6		20	21	−0·014
	11	20	−0·66				
	149	271	+0·00093				

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D. M. Y. SOMMERVILLE.

862. [V. 2. b.] *Madame du Châtelet on Fluxions.*

In view of the long controversy relating to Newton's fluxions, carried on in the *Republick of Letters*, for 1735, 1736, and in the *Works of the Learned* for 1737, by James Jurin, Benjamin Robins and Henry Pemberton, some interest attaches to the interpretation found in the French translation of the *Principia* made by Madame du Châtelet and brought out in 1759 as a posthumous publication. This lady had received instruction in mathematics and Leibnizian philosophy from the Swiss mathematician, Samuel König. Her final rejection of Leibnizian monads and acceptance of Newtonian physics were accompanied by a dispute with König about monads, the infinitely little, and *vis viva*, which their mutual friend Voltaire witnessed with pain. Later Madame du Châtelet had Clairaut as instructor. Her translation of the *Principia* is not literal. The publisher says in the Introduction, " Newton will often be found more intelligible in this translation than in the original and even than in the English translation." One point of dispute of Jurin with Robins and Pemberton was the question, Did Newton's variables reach their limits ? Jurin answered, Yes ; Robins and Pemberton answered, No. Newton's words (*Principia*, 3. Ed., Bk. I., Sec. I., Lem. I.) are : *Quantitates, ut & quantitatum rationes, quae ad aequalitatem tempore quovis finito constanter tendunt, & ante finem temporis illius proprius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo aequales.* Robins gives a literal translation : " Quantities, and the ratio of quantities, that during any time constantly approach each other, and before the end of that time approach nearer than any given difference, are ultimately equal." He maintains that " given difference " means " difference first assigned," that " where the approach is determin'd by a subdivision into parts," " it is obvious, that no coincidence can ever be obtain'd." Jurin, on the other hand, argued that " fiunt ultimo aequales " means " do at last become actually, perfectly, and absolutely equal." Madame du Châtelet renders this passage " deviennent à la fin égales," and in the proof of the lemma, gives the phrasing " leur différence ne sera donc pas plus petite que toute différence donnée," *i.e.* less than *every* given difference. This would seem to place her on the side of Jurin. In the scholium of Bk. I., Sec. I., Lem. XI., she translates freely and refers to " ce que deviennent les quantités, lorsqu' elles atteignent leurs limites." This last quotation places her definitely on the side of Jurin ; the limit is " attained." But this view of a limit did not generally prevail in France ; D'Alembert, writing in Diderot's *Encyclopédie* (1754 and later editions), art. " Limite," does not permit a variable to reach its limits. FLORIAN CAJORI.

University of California.

863. [L¹. 10. d.] *On* Note 848 (*Gazette*, xiii. p. 200) : *Miquel's Theorem.*

With reference to this Note, Mr. H. G. Forder calls my attention to a paper in the *Math. Annalen* (47, p. 564 ; 1896), not mentioned by Coolidge, in which Godt proves the existence of the configuration of 2^n circles that I described. Godt, who refers to Steiner, but not to Clifford or Miquel, devised a symbolical method which one would think was powerful enough to be put to further use. To prove the existence of the configuration is essentially to prove Clifford's theorem. The difficulty in constructing a proof from first principles is to avoid losing one's way as the figure becomes more and more complicated, and in Godt's proof, as in my own, an elastic notation is the guiding thread.

E. H. N.

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405. Louis Antoine de Bougainville, b. 1729, studied in Paris; instead of becoming, as his friends expected, an advocate at the Palais, he enlisted in the Mousquetaires Noirs. Fifteen days later he surprised his friends still more by publishing a treatise on the Integral Calculus. He became an F.R.S. in the same year as Benjamin Franklin.—De Morgan, *Penny Cyclopaedia.*

REVIEWS.

Advanced Calculus. By W. F. Osgood. Pp. xvi + 530. 25s. net. 1925. (Macmillan.)

In any book of this nature, the desire to write with mathematical precision must often clash with the wish to make a large appeal to physicists and chemists, who may be desirous of obtaining in the book a selection of mathematical tools with which to work. Evidences of this conflict are of frequent occurrence in Professor Osgood's book.

The plan of the work is to cover a systematic account of integration, partial differentiation, elementary differential equations and Calculus of Variations. The knowledge so gained is to be applied to such questions as Green's Theorem, Flow of Heat, Vibrations, Hamilton's Principle and Thermodynamics. These applications are, on the whole, carried out with admirable clarity, the sections on Entropy being particularly noteworthy in this respect. But in dealing with the theory forming the groundwork of the book, Professor Osgood is sometimes less happy.

That part of the theory which seems most open to adverse criticism is the treatment of the definite integral. This is defined as the limit of the single Riemann sum, *i.e.*

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x'_k) \, . \, \Delta x_k,$$

and although continuity of the function in all the independent variables is always assumed, no attempt is made to prove that the integral, defined in this way, ever exists, a somewhat notable omission for an author of Professor Osgood's deservedly high reputation. Had some of the semi-geometrical sections, intended to make the treatment appear "plausible," been omitted and replaced by a thorough-going discussion by means of the upper and lower Darboux integrals, the whole of the account of integration would have gained in simplicity and power. Moreover it is probable that the author would then have realised and avoided the confused thinking in some sections, for example, in his treatment of the mass of a lamina of variable density.

An isolated chapter gives an interesting introduction to Vector Analysis, while a valiant attempt to cover the Theory of a Complex Variable, from scratch to Cauchy's and Taylor's Theorems, in thirty pages, is more successful than one might expect.

An annoying feature is the lack of order, though in fairness it must be stated that we are warned in the preface that the order is not that in which the author would teach the subjects concerned. A typical example is that continuity of a function of two variables is used explicitly on p. 45 and succeeding pages, though no definition of this ambiguous term is attempted till p. 107.

Minor blemishes are the use of $\dfrac{dy^2}{dx^2}$ for $\left(\dfrac{dy}{dx}\right)^2$ and the choice of h^2 as a first illustration of the definition of an indefinite quadratic form; misprints are few and unimportant.

This should prove a useful second text-book on Calculus for intending mathematical physicists. T. A. A. B.

Die sokratische Methode und wir Mathematiker. By Dr. Hermann Weinreich. Pp. 50. 2·20 m. 1926. (Berlin, Otto Salle.)

This valuable and interesting essay, by a well-known mathematical teacher and author, describes the new movement in Germany, centred at the University of Göttingen and led by Professor L. Nelson, to combat the dogmatic methods still, in spite of the heuristic revival, too prevalent in education and particularly in philosophy, by a return to the ancient Socratic spirit, the famous maieutics. This young school (*Die Friessche Schule*) springs, *longo intervallo*, from Jakob Fries (1773-1843). In the spirit of Socrates, Fries stressed the scientific method of his master, Kant, rather than the matter of his teaching, and urged the vital need for philosophy to profit by the fruitful method of discovery and verification followed by the physical and mathe-

matical sciences, instead of losing itself in the barren sands of unverifiable speculation. The pamphlet contains a useful bibliography of the new movement (1904-1926), which promises a good harvest. The essay is a supplementary publication of the *Unterrichtsblätter für Mathematik und Naturwissenschaften.* B. B.

An Introduction to Mechanics. By J. P. Clatworthy. Part I.—**Statics.** Pp. x + 207. Part II.—**Dynamics.** Pp. viii + 225. 8s. 6d. 1926. (Methuen.)

This book is divided into two separate parts, each with its own preface and short index, one part dealing entirely with Statics and the other with Dynamics. It is intended for beginners, Science and Engineering students, who wish to obtain some knowledge of Mechanics.

The treatment of Statics is based on the Triangle of Forces, and a large number of examples of various kinds is worked out by this method in the first chapter, after which there follows a short chapter on the Polygon of Forces. Chapter III. opens with some definitions and proofs of moments and couples, and then proceeds to the discussion of the general conditions of equilibrium of a system of forces, based on the force and the couple. Then follow chapters on Centre of Gravity, Friction and Mechanical Work, and the first part concludes with a brief account of some machines. The method of resolving forces is obtained from the Polygon of Forces, and the resultant of two parallel forces is first introduced as a limiting case of the resultant of any two forces: the Lever and problems on parallel forces generally receive a brief treatment at the end of Chapter III.

The first chapter of Part II. gives the formulae for linear motion with constant acceleration, and the next chapter proceeds at once to the "law of acceleration," which is given (after some introductory explanation) in the form $ma = F$, thereby avoiding any mention of Newton's Laws of Motion. Chapters III. and IV. contain some mention of work, energy, momentum and impact; and simple harmonic and pendulum motion are introduced in Chapter V. In the treatment of S.H.M. the expression $x = a \sin \omega t$ is assumed, for the displacement, and expressions for the velocity and acceleration are obtained from this by limiting processes; the projection of circular motion is given only as an illustration. In these first five chapters all the definitions for velocity, acceleration, etc., are for motion in a straight line only, and this has the disadvantages that it is necessary to come back later and consider them afresh, extending the definitions, and to show that the "law of acceleration" holds generally, while the motion of projectiles cannot be considered until after this has been done. On the other hand, the beginner is enabled to get on at once to the easier problems of rectilinear motion, while S.H.M. and normal acceleration can be introduced earlier, and the more difficult problems on impact and relative motion are left until practically the end.

The author is at great pains to be rigorous in his proofs, particularly in the chapter on Centre of Gravity, and he does not hesitate to use limiting processes where necessary, the proofs of $\underset{\phi \to 0}{\mathrm{Lt}} \frac{\sin \phi}{\phi}$ and $\underset{\phi \to 0}{\mathrm{Lt}} \frac{1 - \cos \phi}{\phi}$ being assumed from Trigonometry. In Part I., p. 102, Ex. 1, the positions of the forces on the rod are not stated, and have to be inferred from the figure and the proof; and in Part II., pp. 26-7, the explanation that constant force produces constant acceleration and not constant velocity does not seem to me very convincing. The book contains a very large number of worked-out examples of many different kinds, though the method of working of a few of them appears rather to encourage mere substitution in formulae. J. W. H.

An Introduction to the Theory of Infinite Series. T. J. I'a Bromwich. Second Edition, revised with the assistance of T. M. MacRobert. Pp. xv + 535. 30s. 1926. (Macmillan.)

Mathematical students and lecturers will welcome with pleasure the second edition of Dr. Bromwich's *Infinite Series.* The first edition has been out of print for some years, and was so much appreciated and valued that second-

hand copies were eagerly sought even at enormous prices. It was a deservedly popular book for many reasons. Foremost was its novelty, as no other single book dealt adequately with the numerous topics touched upon, some of which, such as the theory of summability, and the proof of the Euler-Maclaurin summation formula, were not easily accessible to English readers. The book thus made an appeal to those interested in the theory, as well as to those concerned with the applications of the subject. There was also a plenitude of interesting results and striking examples, including many recent ones. Finally, its freshness, stimulating power and suggestiveness exerted a great influence on its readers, many of whom were by its means initiated into modern mathematical ideas and modes of thought.

The second edition consists largely of a reproduction of the first with additional theorems and examples, but the account of the summability of series has been omitted. This omission detracts from the value of the book, as it must from any modern book on infinite series. Some compensation is to be found in the additions, which include a discussion of the solution of linear differential equations of the second order and an account of Gibbs' phenomenon in trigonometric series. One may also mention the Hardy-Landau converse of Cauchy's theorem on limits, but this will not be fully appreciated without a knowledge of summability. Neither will Fejér's theorem on Fourier's series.

Dr. MacRobert's share of the work has been confined to matters of detail, as the entire book except Appendix III. was already in type when it came into his hands. The public will be grateful to him for seeing the book through the press.

Many of the factors contributing to the popularity of the first edition will still apply to the second, for which there also ought to be a considerable demand. There is, however, one very important difference. Dr. Bromwich has taken a prominent part in educating his public to expect more from a book on Infinite Series than they will find in the second edition. The times have changed since 1908 when Williamson and Edwards perhaps still reigned supreme, and much pioneering work had to be done, and new ideas were rather gingerly introduced to students. The modern student realises that a great deal of preparatory work is required for a logical and systematic development of this subject. It may have been all right then to put the arithmetic theory of irrational numbers and limits as Appendix I. Now it is expected in Chapter I., as most classes will soon be informed that irrational numbers must be defined before limits can be discussed.

We should also like to see a short account of the properties of continuous functions, instead of their insertion as examples in Appendix I. Finally, as integrals necessarily play so important a part throughout the book, we should have preferred, even if it had involved a change in the title, an early chapter on the existence, properties and convergence of integrals rather than their inclusion in Appendices II. and III., while a knowledge of the convergence of integrals is required in Chapter II. It seems strange, nowadays, to have a discussion on the existence of the area of the rectangular hyperbola.

L. J. MORDELL.

Readable Relativity. By C. V. DURELL. Pp. vii + 146. 3s. 6d. net. 1926. (Bell.)

Mr. Durell has written for "ordinary people," and "attempts to secure as high a degree of definition as is compatible with the standard of mathematical knowledge of the average person." With this object he has, in general, avoided algebra and replaced it by arithmetic. Unfortunately this makes several of the arguments rather lengthy, and in spite of the elementary nature of the calculations the reader requires a good deal of staying-power to follow them through to the end. Each chapter is followed by a number of examples, chiefly numerical, and a complete set of answers is given. Every effort has been made to brighten the subject, and there are some humorous quotations from *Alice through the Looking-glass*. A portrait of Einstein appears as the frontispiece.

The first chapter and the last are very readable, and they are no more difficult than the scientific articles that appear at intervals in certain weekly

papers. The other eight chapters, however, call for serious study. It is not so much that any one step of the argument is difficult in itself, but rather that concentration is required to follow a succession of such steps without losing sight of the cumulative effect of the whole. We get some algebra on pp. 110-111. This is really very easy, but it will require patience from those who have let their algebra rust. The geometry on pp. 94-96 is more difficult. It is to be feared that the author has over-estimated both the mathematical knowledge and the perseverance of the " average person " for whom the book is intended.

It is only fair to say that Mr. Durell has taken a great deal of trouble in presenting his subject. He has included some account of recent results, such as the much-discussed work of Dayton Miller and Eddington's objection to the hypothesis, based upon it, of an Ether-Drift increasing with the distance from the surface of the earth. The book may not have succeeded in removing the great difficulties of the Theory of Relativity, but it will arouse interest, and by its low price it will be accessible to a wide circle.

Probleme der Atomdynamik. By M. Born. Pp. viii + 183. Paper covers 10·50 Reichsmarks, bound 12 Reichsmarks. 1926. (Springer.)

This work is of exceptional interest, as it contains (among other things) what is probably the first account in book-form of a new and important branch of Mathematical Physics. This new science may be called Atomic Mechanics, or the new Quantum Mechanics. It must not be confused with the usual form of Quantum Theory associated with the names of Bohr and Sommerfeld. The fundamental ideas of Atomic Mechanics were published by Heisenberg of Göttingen as recently as 1925. They were developed by Born and Jordan, also of Göttingen, and, from a different point of view, by Dirac of Cambridge. The work of these pioneers, with its application of a somewhat abstruse branch of Pure Mathematics to the complicated phenomena of Spectroscopic Physics, rapidly attracted attention. It has a unity of structure and clarity of principles lacking in the older Quantum Theory, and it leads, without ambiguity or special hypotheses, to numerical results which are in accurate agreement with those founded by experiment. New researches are continually appearing in England, as well as in Germany, and any account of these is almost bound to become out-of-date in the interval between writing and publication. However, it appears desirable to attempt to give the readers of the *Gazette* an outline of the principles and objects of this new science. Many important points will have to be passed over, but it is hoped that what is given will be sufficient to enable readers to realise the great importance of the subject and to arouse the desire for a fuller account, such as that in the book under review.

Like Bohr's Quantum Theory, the new theory deals with the frequency and intensity of spectral lines. The old theory tried to set up a series of rules by which these frequencies and intensities could be deduced from certain supposed orbital motions of one or more electrons round a nucleus. The new theory ignores these motions, which there is little hope of ever observing, and deals directly with the frequencies and intensities themselves. Now a well-known experimental result, the *Rydberg-Ritz Combination Principle*, states that every frequency ν of a spectral line is of the form $T_n - T_m$, where n and m are positive integers. T_n is the same function of n as T_m is of m. To indicate its dependence on n and m, we may write the frequency as $\nu(nm)$, and we see, from the values in terms of the T's, that

$$\nu(mn) = -\nu(nm) \quad \text{and} \quad \nu(nk) + \nu(km) = \nu(nm).$$

Denoting by $C(nm)$ and $\delta(nm)$ respectively the intensity and phase corresponding to $\nu(nm)$, and using t for the time, we have to consider phenomena represented by a doubly-infinite set, each member of which is of the form $C(nm)\cos\{2\pi\nu(nm)t + \delta(nm)\}$. It is more convenient to deal with the set of forms $q(nm)e^{2\pi i\nu(nm)t}$, which can be examade ctly equivalent if $q(nm)$ and $q(mn)$ are conjugate complex numbers, each of modulus $C(nm)$, for then

$$q(nm)e^{2\pi i\nu(nm)t} + q(mn)e^{2\pi i\nu(mn)t}$$
$$= C(nm)\cos\{2\pi\nu(nm)t + \delta(nm)\} + C(mn)\cos\{2\pi\nu(mn)t + \delta(mn)\},$$

provided that $C(mn) = C(nm)$, $\delta(mn) = -\delta(nm)$, and, as before, $\nu(mn) = -\nu(nm)$. The aggregate of all such forms may be written down in rows and columns, as follows:

$$\begin{array}{llll} q(11)e^{2\pi i\nu(11)t}, & q(12)e^{2\pi i\nu(12)t}, & q(13)e^{2\pi i\nu(13)t}, & \ldots \\ q(21)e^{2\pi i\nu(21)t}, & q(22)e^{2\pi i\nu(22)t}, & q(23)e^{2\pi i\nu(23)t}, & \ldots \\ q(31)e^{2\pi i\nu(31)t}, & q(32)e^{2\pi i\nu(32)t}, & q(33)e^{2\pi i\nu(33)t}, & \ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots \end{array}$$

Such a set of forms, rather like the set of constituents of a determinant, is called a *Matrix*. It is not the most general type of matrix, because of the relations between $q(nm)$ and $q(mn)$ and between $\nu(nm)$ and $\nu(mn)$. For brevity, the matrix may be denoted by a single symbol $\mathbf{q}$, and any symbol in this heavy type must be understood to refer to a matrix. A matrix is dealt with in Atomic Mechanics very much as a single variable, such as a displacement, is dealt with in ordinary mechanics.

We have now to formulate the laws concerning these matrices. It is natural to take over from the pure mathematicians their rules for addition, subtraction, and multiplication. The sum of two matrices is defined as the matrix whose constituents each consist of the sum of the two corresponding terms of the given two matrices. Similar obvious definitions are given of subtraction and of differentiation with respect to the time. The definition of multiplication is more complicated. $\mathbf{qp}$ is defined as the matrix whose typical constituent is $\Sigma q(nk)e^{2\pi i\nu(nk)} \cdot p(km)e^{2\pi i\nu(km)}$, where the summation is for all integral values of k. This expression simplifies in virtue of the relation $\nu(nk) + \nu(km) = \nu(nm)$, so the frequencies which occur in the product are exactly the same as those occurring in the factors, which is of great importance if the product is to have any physical meaning. [Of course, the identity of the frequencies in $\mathbf{q}$ and $\mathbf{p}$ is assumed here, from physical reasons. In pure mathematics we omit these exponential factors, for either the constituents are supposed independent of the time or the time factor may be supposed absorbed with the rest into a single symbol.]

Another natural step to take in setting up a new form of mechanics is to seek help from ordinary dynamics in its most general form, namely that due to Hamilton. Now Hamiltonian dynamics, as applied to a system of one degree of freedom, depending on a single coordinate q, takes the corresponding momentum, usually denoted by p, as an additional variable. It then deals with a certain function H (the *Hamiltonian function*), which is quadratic in p and q. (In most cases H is simply the sum of the kinetic and potential energies, expressed in terms of the momentum and displacement.) The equations of motion then become

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p},$$

which are generally called the *Canonical Equations*.

We endeavour to take over these equations with the least possible change, merely replacing the p, q, and H by the matrices $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{H}$. But how are we to interpret differentiation of one matrix with respect to another? This is one difficulty of the matrix calculus. Another is that multiplication, as defined above, is *non-commutative*. These two difficulties are both dealt with in a way that appears, at first sight, extremely unnatural, namely:

$$\frac{\partial \mathbf{H}}{\partial \mathbf{p}} = \frac{2\pi i}{h}(\mathbf{Hq} - \mathbf{qH}), \quad \frac{\partial \mathbf{H}}{\partial \mathbf{q}} = \frac{2\pi i}{h}(\mathbf{pH} - \mathbf{Hp}), \qquad \text{(definitions)}$$

$$\mathbf{pq} - \mathbf{qp} = \frac{h}{2\pi i}\mathbf{1}, \quad \text{(postulate, usually called } \textit{Die Vertauschungsregel}\text{)}.$$

The $\mathbf{1}$ denotes a matrix all of whose constituents are either unity (for $n = m$) or zero (for $n \neq m$). The h is Planck's constant. *Die Vertauschungsregel* may be translated *The Commutative Rule*, though perhaps *The Non-Commutative Rule* would be more appropriate. Of course, $\mathbf{p}$ and $\mathbf{q}$ are not *any* two matrices. They are related in a definite way, like momentum and displacement.

These definitions and this postulate are not so artificial as they look. They were really obtained from somewhat elaborate physical and dynamical considerations, too lengthy to be given here. What is more, the work of Dirac leads to the conclusion that no other definitions are really possible, if the work is to retain any significance. By adopting these we obtain the *Canonical Equations*,

$$\dot{\mathbf{p}} = \frac{2\pi i}{h}(\mathbf{Hp} - \mathbf{pH}), \quad \dot{\mathbf{q}} = \frac{2\pi i}{h}(\mathbf{Hq} - \mathbf{qH}).$$

The left-hand sides of these equations may be transformed, introducing a *Diagonal Matrix* $\mathbf{W}$, defined by $W(nm) = 0 (n \neq m)$, $W(nn) = hT_n$ (where h is Planck's constant, and $\nu = T_n - T_m$). The final forms are :

$$\mathbf{Wp} - \mathbf{pW} = \mathbf{Hp} - \mathbf{pH}, \quad \mathbf{Wq} - \mathbf{qW} = \mathbf{Hq} - \mathbf{qH}.$$

We have now enunciated the principles of Atomic Mechanics for a system of the simplest kind. In a more complicated case (corresponding to an ordinary dynamical system of two or more degrees of freedom), it might be thought that a doubly-infinite aggregate would have to be replaced by a quadruply-infinite one or worse. However, it is easy to deal with these cases by a two-dimensional matrix of the ordinary kind.

Wiener has shown that in the case of *Aperiodic Motions* the matrices must be replaced by operators of somewhat similar form. There are many other developments of the subject, such as the theory of perturbations and the use of contact transformations, that cannot be explained here in detail. A large number of the ideas of Hamiltonian dynamics can be applied almost unchanged.

It has been claimed that one of the first successes of the new theory was the detailed explanation of the complicated cases of the *Zeeman effect* (the splitting up of a single spectral line into many by the application of a magnetic field). However, an essential part of this explanation consisted of Uhlenbeck and Goudsmit's new conception of an electron, as not merely a point-charge, but an entity with angular momentum and magnetic moment. But no such hypothesis was needed in explaining the *Compton effect* (the change of wave-length and direction produced when X-rays and γ-rays are scattered by impact). The theory has also been applied to the explanation of *band spectra*. It is highly probable that a good deal more will have been accomplished by the time that this review is published.

The new Quantum Mechanics, with a preliminary account of the older theory and of Hamiltonian dynamics, forms the subject-matter of twenty out of the thirty lectures of which Prof. Born's book consists. The other ten deal with the properties of crystals, and they present a wonderful combination of mathematics, physics, chemistry and crystallography. This part of the book will be dealt with in a future number of the *Gazette*.

To conclude, it may safely be said that no one who desires to keep in touch with the latest developments in Mathematical Physics can leave Prof. Born's work unread.

H. T. H. PIAGGIO.

The Geometry of René Descartes, translated from the French and Latin. By DAVID EUGENE SMITH and MARCIA L. LATHAM. With a facsimile of the first edition, 1637. Pp. xiii + 246. $4.00. 1925. (Chicago, London : The Open Court Publishing Company.)

Mathematicians will be grateful to the Open Court Publishing Company for including the *Géométrie* of Descartes in their series, and so making accessible in a convenient form a classic which it is probably difficult for an ordinary student to come by, and which is therefore too little known. The present edition gives us, with an English translation and notes, an actual facsimile of the first edition of the work, which appeared in 1637, not independently, but as the final part of a volume containing the *Discours de la Méthode*, the *Dioptrique*, the *Météores* and the *Géométrie*. It occupied pages 297 to 413 of the volume, and the facsimile, of course, retains this paging.

Descartes (1596-1650), philosopher and mathematician, is universally known as the inventor of the method of coordinates in geometry, whence the name Cartesian Coordinates. There had, of course, been anticipations of the idea in ancient times. Thus Apollonius of Perga's statement of the fundamental

property of each of the three conics is the exact equivalent of the Cartesian equation referred to any diameter and the tangent at its extremity as coordinate axes. Apollonius, too, in his treatise on *Plane Loci* stated in words the equivalent of the fact that the locus of a point the coordinates of which satisfy an equation of the first degree is a straight line. But the ancient Greeks had not any notation, except the parts of some geometrical figure, to represent quantities other than numbers. What Descartes did was to use the symbolism of algebra (then recently introduced into France from Italy), and to apply all the resources of algebraical calculation, in aid of purely geometrical reasoning, for the purpose of determining the parts of the geometrical figures required in constructions. His method thus had the momentous effect of removing once for all the disabilities which prevented Greek geometry from making any further advance beyond the stage to which it had been brought by Eudoxus, Euclid, Archimedes and Apollonius.

Not that Descartes is alone entitled to the credit of this revolution in method. If it is a question of priority of discovery, a good claim could be put forward on behalf of Descartes's great contemporary, Fermat. It is true that Fermat's work, *Ad locos planos et solidos isagoge*, was not published till much later (1697); but it was certainly conceived and perhaps written before 1637, the date of publication of the *Géométrie*; moreover, the method of coordinates comes out much more clearly in Fermat, and Fermat's analytical geometry generally is much more like ours than is Descartes's.

Anyone who turns to the *Géométrie* with any expectation of finding even the makings of an introduction to coordinate geometry such as is supplied in our ordinary text-books will be disappointed. Nothing could well be more different; indeed, as Professor Loria has remarked in a recent study, there is a greater gulf between Descartes's work and a modern treatise on analytical geometry than there is between an ancient (*i.e.* Greek) and a modern treatise on any other mathematical subject. Descartes does not use his coordinates for the purposes of elementary geometry, the geometry of the straight line and circle; he uses them mainly for the investigation of curves higher than those of the second degree, though he devotes about eleven pages of Book II. to elucidating, by actual construction, the nature and position of a conic the equation of which he has found (referred to coordinate axes which are in general oblique), according as we give different values and different signs to the various constants appearing in the equation. It is strange that the usefulness of the method of coordinates for elementary geometry does not appear to have occurred to any one for a long time. The first systematic exposition of elementary analytical geometry as we know it seems to have been given by S. F. Lacroix as part of his great *Traité du calcul différentiel et du calcul intégral* (Paris, 1797).

But it is time to see how the coordinates are actually introduced by Descartes. Book I. of the treatise begins with some generalities, showing the relation between the operations of arithmetic and geometry; how, for instance, if we choose a particular line as the unit, the operation of multiplying or dividing quantities can be performed by finding a fourth proportional, and the extraction of the square root by finding the mean proportional. Descartes next observes that, if we wish to solve any problem, we must first consider it done and give names to all the lines which appear necessary for the construction, those which are unknown as well as the others. Then, without making any distinction between those which are known and those which are unknown, we must study closely all the conditions, following such an order as will reveal, in the most natural manner, the way in which the several lines depend one upon the other, until we have found means of expressing one and the same quantity in two different ways; the result is called an equation. And we should try to find as many equations as there are unknown lines according to the assumptions made. If there are not so many equations, notwithstanding that we have not omitted to take account of a single one of the conditions of the problem, this shows that the problem is not entirely determinate, in which case we can arbitrarily take lines of known length in place of those to which there is no equation that corresponds, and so on. Then, if there are several equations, we must use each in order, either considering it alone or comparing

it with the others, until we find an equation in one unknown, *e.g.* $z=b$, $z^2=-az+b^2$, $z^3=az^2+b^2z-c^3$, etc.

After giving a geometrical solution of three types of quadratic equation, Descartes passes to the Problem of Pappus, which he quotes in a Latin translation with Pappus's explanation. The problem may be stated thus. If from a point straight lines be drawn to meet, at given angles respectively, each of $2n$ or $2n+1$ straight lines given in position, and if (in the first case) the product of the lengths of n of the straight lines so drawn bears a given ratio to the product of the lengths of the remaining n straight lines, or if (in the second case) the product of the lengths of $n+1$ of the straight lines bears a given ratio to the product of the lengths of the n remaining straight lines and a straight line of given length, it is required to find the locus of the point from which all the straight lines are drawn. Pappus explains that, if the given lines are three or four in number, the locus is a conic section; if they are more than four, the locus is a higher curve. He adds that the nature of these higher curves had not been established, though one such, and that not the simplest, had been investigated. Descartes begins by drawing a figure in which there are four fixed lines (no two being parallel). Two of them, say AB, AD, intersect at A. C is the point the locus of which is required. The line drawn from C to meet AB at a given angle meets it at B. Descartes then says: "First I consider the thing already done and, in order to extricate myself from the confusion of all these lines, I consider *one* of the given lines and *one* of those which have to be found, for instance AB and CB, as the principal lines and those to which I shall try to refer all the others. Let the segment of the line which is between the points A and B be called x, and let BC be called y." In this unobtrusive way he introduces his coordinates as "principal lines"; he does not call them "coordinates," or use the word "axis"; he does not even draw what we should call the axis of y (the line through A parallel to BC). He then produces all the other lines in the figure to meet the "principal lines" (produced if necessary), and shows that each of the lines drawn from C to meet at given angles the given fixed lines other than AB can be determined as linear expressions in x, y and constants. Descartes further observes that in the product of any number of these lines when multiplied together the degree of any term containing x or y cannot be greater than the number of the lines multiplied together. This determines in general the degree of the equation in x and y which represents the locus required.

The only other passage in which Descartes states the general principle of representing a curve by an equation seems to be in Book II. (p. 319), where he says that all the points on curves "which can be called geometric, *i.e.* which fall under some precise and exact measure," have necessarily some relation to all the points of one straight line which can be expressed by some equation, one and the same equation covering all the points on the curve. (The one straight line is what we may call the axis of x, and "all the points" of the line are the feet of the ordinates (y) drawn from each point on the curve to meet it at a given angle; each ordinate (y) determines a certain abscissa (x) measured along the one straight line from a fixed point on it.)

Book II. begins with a criticism of the objections raised by the Greeks to what they called "mechanical" constructions (constructions by means of machines more complicated than the ruler and compasses). Provided that such a construction is effected by some continuous motion, or by a succession of such motions where each is completely determined by that which precedes it, Descartes sees no objection, and he describes two such means of constructing curves. The first is a kind of glorified mean-finder, reminding us of Eratosthenes's and other machines for constructing two or more mean proportionals. It produces a succession of curves for which Descartes gives no equations, but which we may represent in polar coordinates by $\rho=a/\cos^{2m}\omega$. The second uses a curve moving along, with its axis always lying on a fixed straight line, and a ruler always held so as to pass through two points, one of which is a point on the axis of the curve at a fixed distance from its vertex, while the other is a fixed point lying neither on the curve nor on its axis; the intersection of the curve and the ruler at any moment during the motion determines a point, and the locus of these points is a certain curve. If for

the moving curve we substitute a triangle, and the triangle moves along the bisector of an angle (a fixed point on this bisector taking the place of the fixed point on the axis of the curve), the locus is a hyperbola the equation of which Descartes gives. If the moving curve is a circle and the fixed point on its diameter is the centre, the locus is a conchoid. If the moving curve is a parabola, the locus is a curve represented by an equation of the third degree; Descartes later finds its equation, in a particular case, to be

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy,$$

where a is the latus rectum of the parabola, and proves that precisely this curve serves to solve a particular case of Pappus's problem, that, namely, in which four of the five given lines are parallel and are separated by equal distances (a), while the fifth cuts them at right angles, and, moreover, they are all met at right angles by the lines drawn from the point the locus of which is required.

The next section gives an original method of finding the normal to a curve at a given point. Descartes observes that, assuming the normal to be drawn at a point C and to meet the axis at P, the circle with P as centre and passing through C will touch the curve (instead of cutting it) at C. If CM be the perpendicular from C to the axis AP, and A be the vertex, Descartes puts $y = AM$ (the abscissa), $x = CM$ (the ordinate), $v = AP$, $s = CP$ (the normal), then $MP = v - y$ and $s^2 = x^2 + (v - y)^2$. Eliminating x between this equation and the equation of the curve, we have an equation in powers of y. If CP is the normal, this equation must have two equal roots (e, say), and hence the expression equated to zero must have $(y - e)^2$ as a factor. If, for example, the equation is $y^6 - by^5 + cy^4 - dy^3 + fy^2 - gy + h = 0$, Descartes assumes that this is equivalent to

$$(y - e)^2(y^4 + py^3 + q^2y^2 + r^3y + t^4) = 0;$$

then, equating coefficients, we have six equations to enable us to determine p, q, r, t, s and v; and either s or v with e (or y) gives the normal required. It is then shown that the same method is possible with a curve represented, not by an equation in x, y, but by a relation between the distances of any point on the curve from two fixed points respectively, such as the first of Descartes's ovals, curves of the type which we should represent by equations such as $ar_1 + br_2 = \text{const.}$ Then, after a remark that he does not add the constructions for the normals or tangents at points on a curve obtained by an algebraical calculation of this kind, because they are easy, although requiring a certain address if they are to be made short and simple, Descartes gives a very simple construction for the normal at a point on Nicomedes' conchoid, which it is difficult to believe that he obtained by the method of algebraical calculation. He does not say how he obtained it, but it must evidently have been by considering the instantaneous direction, at the moment, of the motion of the point describing the curve, and resolving the motion into two components in the same way as Archimedes must have determined the direction of the tangent to his spiral at any point, *i.e.* by a sort of anticipation of the differential calculus. Book II. ends with a description of four different "Cartesian ovals," which are useful for Descartes's theory of Catoptric and Dioptric as furnishing curves which will reflect and refract rays in accordance with given relations between angles of incidence and reflection or refraction respectively.

Book III. ("On the construction of problems which are solid or more than solid") is mainly on the solution of equations of the third and higher degrees. Descartes first lays down certain principles. Every equation should have as many roots as it has dimensions, but the roots may be "false" (by which he means negative). If $x = a$ is a root, $x - a$ is a factor of the expression equated to zero. An equation can be transformed into another in which the roots are increased or diminished by any given amount; by this means we can always get rid of the second term in an equation; or we can (1) make all the roots positive ("true") and (2) make the coefficient of the third term greater than the square of half the coefficient of the second. We can also multiply or divide the roots (without knowing them) by any given quantity. When an equation is reduced to a cubic, we first try, by means of the factors of the absolute term, to find a linear factor $x - a$. If we succeed, we have a solution,

and further division by $x-a=0$ leaves a quadratic equation. If not, the problem is "solid." In the case of a bi-quadratic equation, we try in the same way to find a linear factor. If we can, this may give the solution required; if not, we have a cubic equation left, which we may examine in the same way. Alternatively, the biquadratic expression equated to zero may be capable of being split into two quadratic factors. If the equation is $x^4 \pm px^2 \pm qx \pm r=0$, we must, says Descartes (without, for the moment, any explanation), write $y^6 \pm 2py^4+(p^2 \mp 4r)y^2-q^2=0$. What he really means is that we must assume that the original equation can be expressed as

$$(x^2-yx+z)(x^2+yx+v)=0;$$

whence, by equating coefficients, we can find v and z in terms of the coefficients of the original equation and, substituting these values, we have left the above cubic equation in y^2. As before, we must first try to find a linear factor, as $y^2 \pm a$, of the expression equated to zero. If this is not possible, the problem is "solid."

When the given equation is of the third or fourth degree, a solution can always be found by means of *one* conic with straight lines and circles (if there is a real solution). Descartes solves, by means of a *parabola* and circles, equations of the forms $z^3= \pm apz \pm a^2q$ and $z^4= \pm apz^2 \pm a^2qz \pm a^3r$. Then he solves, in the same way, the problem of the two mean proportionals, equivalent to $z^3=a^2q$, and that of the trisection of any angle, equivalent to $z^3=3z-q$. Next, quoting Cardano's solution of the equation $z^3= \pm pz+q$, he puts forward his construction for trisecting an angle as an alternative method. Then, after some remarks on the "irreducible case," he goes on as follows: "You know already how, when we seek the quantities required for the construction of these problems, we can always reduce them to some equation of degree not higher than the sixth ('square of cube') or the super-solid. Again, you also know how, by increasing the value of the roots of this equation, you can always make them all true (*i.e.* positive) and also ensure that the coefficient ('known quantity') of the third term is greater than the square of half that of the second term; and lastly, how, if it is not of higher degree than the sixth ('super-solid'), you can always raise it to that degree and ensure that none of the terms are wanting. Now, in order that all the difficulties now in question may be solved by one and the same rule, I assume all these things done and the reduction made to an equation of the form

$$y^6-py^5+qy^4-ry^3+sy^2-ty+u=0,$$

where the quantity called q is greater than the square of the half of that which is called p."

He then actually solves this equation by means of a curve of the before-mentioned type, namely that described by the intersection (1) of a parabola with principal parameter

$$\sqrt{\frac{t}{\sqrt{u}}+q-\frac{1}{4}p^2} \quad (=n,\ \text{say})$$

moving along its axis, and (2) of a ruler moving so as always to pass through (a) the point on the axis of the parabola the distance of which from the vertex is $2\sqrt{u}/pn$, and (β) a point in the plane in which the parabola moves such that the length of the perpendicular from the point to the axis is equal to $\frac{1}{2}p$. He gives no hint of how he arrived at his construction, but the curves which by their intersection actually give the solution are (1) the curve constructed in the way just described, and (2) a certain circle. If we take as the axis of x the straight line along which the axis of this parabola slides, and as the axis of y the line perpendicular to it which passes through the centre of the circle, the curve is

$$nxy-y^3+\tfrac{1}{2}py^2+\frac{ty}{2\sqrt{u}}-\sqrt{u}=0,$$

and the circle is
$$x^2+y^2-2\frac{m}{n^2}y=\frac{t^2}{4n^2u}-\frac{s}{n^2}-\frac{p\sqrt{u}}{n^2},$$

where
$$n=\sqrt{\frac{t}{\sqrt{u}}+q-\tfrac{1}{4}p^2}$$

and
$$m=\tfrac{1}{2}r+\sqrt{u}+\frac{pt}{4\sqrt{u}}.$$

But Descartes does not write down any one of these equations as such. He simply draws the curves, and then proves that their common points give the solution of the given equation. His verification of the fact is the equivalent of eliminating x between the two equations given above. In other words, Descartes's work is not really coordinate geometry in our sense, but rather a continuation and extension (with the aid of algebra) of Greek investigations and methods ; he leaves us as much in the dark about the form of his analysis as do the Greeks about theirs.

As a special case of the last problem, Descartes shows how to find four mean proportionals between a and b. For, taking the equation $x^5 - a^4b = 0$ or $x^6 - a^4bx = 0$, we transform it, by putting $y - a = x$, into

$$y^6 - 6ay^5 + 15a^2y^4 - 20a^3y^3 + 15a^4y^2 - (6a^5 + a^4b)y + a^6 + a^5b = 0.$$

He adds that by the same method we can divide any angle into five equal parts, or inscribe in a circle a regular polygon of 11 or 13 sides.

The book is not easy to read ; indeed, Descartes seems purposely to have left many things obscure or only half explained. He makes out that this is in the interest of his readers, in such sentences as the following : "I shall not stop to explain this more in detail, because (by so doing) I should rob you of the pleasure of learning it by yourself, and the utility of cultivating your mind by exercising yourself in it" (p. 301). "For the rest I have here omitted the proofs of the greater part of what I have said, because they seemed to be so easy that, provided you take the trouble to examine methodically if I have been wrong, they will automatically present themselves to you ; and it will be more useful to learn them in this fashion than by reading them" (p. 389). "I hope that posterity will give me credit, not only for the things which I have explained, but also for those which I have voluntarily omitted in order to leave them the pleasure of discovering them" (*ad fin.*). "This is no more difficult than what I have just explained, or rather it is much easier since the way to it is now open. But I prefer that others should seek the solution, in order that, if they have some little further trouble in finding it, this fact will cause them to appreciate so much the more the discovery of the things here demonstrated" (p. 368). We may, perhaps, detect a truer motive in this last quotation. At a time when researchers were extraordinarily active, and everyone was keen to get personal credit, as against his rivals, for new discoveries, Descartes did not want to make it too easy for anyone else to uplift any idea from his book and proclaim it as a discovery of his own.

A word may be added about Descartes's notation. He set the fashion of appropriating the last letters of the alphabet (especially x, y) for expressing unknown quantities. In expressing powers of quantities he writes, as we do, x^3, y^4, etc. ; but for the square he generally prefers to repeat the symbol, *e.g.* aa $(=a^2)$, xx $(=x^2)$. The square root is $\sqrt{}$, and the cube root is $\sqrt{}C.$; a long line at the top serves to bracket the terms coming under the $\sqrt{}$. + and − are used, though the printer generally prints the latter in two bits thus, − −. A simple dot between two expressions indicates that the sign between them may be either *plus* or *minus*. Equality is represented by ∞, which is like the sign used in our text-books of algebra for "varies as," but is turned the opposite way.

While, as we have said, the Open Court Publishing Company are to be congratulated on including this classic in their series, we could wish that the translation had been better done ; for we have found it in many places inaccurate, and, in some, completely misleading. There is a fair quantity of notes, which are on the whole useful.

T. L. H.

The Dial Machine. By T. C. J. ELLIOTT. 4s. 6d. 1926. (Peterborough Press.)

The pupil is apt to be vague about the meaning of the concepts of mathematics, and any attempt to broaden and deepen his understanding of them deserves all possible encouragement. That is the aim of the Dial Machine, and the book contains many ideas well suited to this aim. We do not doubt that Mr. Elliott's pupils have gained greatly from his ministrations.

But any other teacher will have to work hard for what he gets out of the book. Mr. Elliott has difficulty in conveying his ideas to other people. The exposition is like that of a crossword puzzle. Amendment is necessary. If Mr. Elliott finds it possible to improve his exposition so far as to make it immediately intelligible, there may well be a good future for his little book.

D. B. M.

The Scientific Construction of the Regular Heptagon with Angles correct to Ten Seconds, etc., etc. By T. Alexander. Pp. 8. n.p. 1925. (Ponsonby and Gibbs, Dublin.)

The kernel of this brochure is contained in the problem of "placing in a semi-circle a triangle with its acute angles in the ratio of 3 to 4."

The remaining interest either arises from or underlies this problem; but its presentation is not so happy, being rather complicated and over-elaborate.

The author's solution is neat, and we believe new:

Take a semi-circle VNJ on a diameter VJ. Draw HJ making an angle of 15° with VJ, and equal to $\frac{1}{2}VJ$. Bisect HJ in T and draw TMN perpendicular to HJ, and cutting the diameter in M and semi-circle in N. Join VN, NJ. Then, if $7\beta = 90°$, angle $NVJ = 3\beta$, and angle $NJV = 4\beta$.

This can easily be shown a very close approximation, and is quite worth making a note of. C. H. Chepmell.

YORKSHIRE BRANCH.

The autumn meeting of the Yorkshire Branch of the Mathematical Association was held on Oct. 16 in the Staff House of the University of Leeds—Professor S. Brodetsky presiding.

Thirteen new members were enrolled. Mr. R. W. Evans (Ilkley Grammar School) and Miss E. Green (Carlton Street Secondary School for Girls, Bradford) were elected vice-presidents; Miss I. Greenwood, Miss L. Watson, and Mr. J. H. Everett were elected to the committee; and Mr. A. B. Oldfield (Pudsey) to the Scholarships Committee. Dr. Billen (Leeds Grammar School) was congratulated on his appointment as head master of Ellesmere College, Shropshire.

A paper was read by Dr. J. M. Crofts (secretary to the Northern Universities Joint Matriculation Board) on "The Statistics of Examination Results, with particular reference to external school examinations," and Dr. G. N. Watson (Professor of Mathematics in the University of Birmingham) gave a paper on "Inequalities."

The Chairman announced that Professor H. Lamb and Professor E. T. Whittaker had consented to take part at the meeting to be held at Grantham in March, 1927, on the 200th anniversary of the death of Sir Isaac Newton.

406. William Gifford (b. 1757) was apprenticed at 15 to a shoemaker at Ashburton. Mathematics at first were his favourite study, and he tells us that in the want of paper he used to hammer scraps of leather smooth, and work his problems on them with a blunt awl. He lived to write *The Baeviad* and *The Maeviad*, and to be the first editor of *The Quarterly Review* (1809-1824).

407. To Osiander, the printer of Nuremberg, we owe both the title-page and the preface of the *De Revolutionibus Orbium Coelestium*, 1543—the book that was but seen and touched by the dying Copernicus. De Morgan notes the *Igitur eme, lege, fruere*, on the title-page as the only "puff" of the kind he had ever seen.—"Copernicus," *Penny Cyclopaedia*.

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THE MATHEMATICAL ASSOCIATION.

(*An Association of Teachers and Students of Elementary Mathematics.*)

"*I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto.*"—BACON (Preface, *Maxims of Law*).

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.

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